Spectral resolutions in effect algebras

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Dedicated to the memory of David J. Foulis

Effect algebras

An effect algebra is a system $(E, 0, 1, \oplus)$, where $0, 1 \in E$ are constants, $\oplus$ is a partial binary operation on E such that:

(E1) if $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$;

(E2) if $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $a \oplus (b \oplus c)$ is defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$;

(E3) for every $a \in E$ there is unique $a^\perp \in E$ such that $a \oplus a^\perp = 1$;

(E4) if $a \oplus 1 \in E$, then $a = 0$.

Covers many different algebraic structures: MV-effect algebras, OMPs, orthoalgebras, etc.

Foulis & Bennett, 1994

Hilbert space effect algebras

Effect algebras are an algebraic model of Hilbert space effects:

$$E(\mathcal{H}) = \{E \in B(\mathcal{H}), \quad 0 \leq E \leq I\}$$

- measurements on a quantum system in the Hilbert space formalism
- important special property - spectrality:

  for $a \in E(\mathcal{H})$ there is a family $\{p_{a,\lambda}\}_{\lambda \in [0,1]}$ of projections such that

  $$a = \int \lambda dp_{a,\lambda}$$

Spectral resolution in Hilbert space effect algebras

Assume $\dim(\mathcal{H}) < \infty$.

Any effect $a \in E(\mathcal{H})$ has eigenvalues $\lambda_i \in [0,1]$, with eigenprojections $p_i \in P(\mathcal{H})$. Then

$$p_{a,\lambda} = \sum_{i,\lambda_i \leq \lambda} p_i$$

and we have

$$a = \int \lambda dp_{a,\lambda} = \sum_i \lambda_i p_i .$$

Spectral resolution in Hilbert space effect algebras

Let $a \in E(\mathcal{H})$. The spectral resolution of a is the unique family of projections $\{p_\lambda\}_{\lambda\in[0,1]}$ such that

- $1 = p_1 \geq p_\lambda \geq p_\mu$ for $1 \geq \lambda \geq \mu$ (nondecreasing),
- $\bigwedge_{\lambda>\mu} p_\lambda = p_\mu$ (right continuous),
- $p_\lambda a = a p_\lambda$ (commutativity),
- $p_\lambda a \leq \lambda p_\lambda$, $p_\lambda^\perp a \geq \lambda p_\lambda^\perp$.

Further, a is uniquely determined by $\{p_{a,\lambda}\}$ and a commutes with b iff $p_{a,\lambda}$ commutes with $p_{b,\mu}$ for all λ and μ.

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Question

Can we have some type of a spectral resolution for an abstract effect algebra E?

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What are the additional structures and/or properties of E needed to obtain this?

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Spectrality in effect algebras?

For convex effect algebras:

- spectral duality in order unit spaces (Alfsen-Schultz, 1976, 2003)
- contexts - finite resolutions of the unit (Gudder, 2019; AJ & Plávala, 2019)

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We will work in general effect algebras, using an approach started in (Gudder, 2006; SP, 2006), inspired by spectrality in partially ordered unital abelian groups (POUAG) (Foulis, 2003-2005).

Compressions and compression bases in effect algebras

Let E be an effect algebra.

A compression is an additive map $J : E \to E$ such that

$$a \leq J(1) \iff J(a) = a, \qquad a \leq J(1)^{\perp} \iff J(a) = 0.$$

Gudder, 2006; SP, 2006

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Properties:

- J is idempotent
- focus of J: $J(1)$ is a principal element:

$$a, b \leq p, \quad \exists a \oplus b \implies a \oplus b \leq p.$$

- J has a supplement J': $\mathrm{Im} J = \mathrm{Ker} J'$, $\mathrm{Im} J' = \mathrm{Ker} J$.
- $J'(1) = J(1)^{\perp}$.

Gudder, 2006; SP, 2006

Compressions and compression bases in effect algebras

A compression base: $\{J_p\}_{p \in P}$

- a collection of compressions
- $J_p(1) = p$, for all $p \in P$
- $P \subseteq E$ a subalgebra
- if $p, q \in P$, $p \leftrightarrow q$ (Mackey compatible), then

$$\exists r \in P \text{ such that } J_p J_q = J_q J_p = J_r.$$

Elements of P are called projections.

Gudder, 2006; SP, 2006

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- bicommutant of a:

$$P(a) = \{p \in P : p \leftrightarrow a, \ \forall q \in P, q \leftrightarrow a \implies q \leftrightarrow p\}.$$

a Boolean subalgebra in P

Examples of compression bases

Any effect algebra E can be endowed with compression bases that can be ordered by inclusion:

- Trivial (minimal): $\{J_0 = 0, J_1 = \mathrm{id}\}$.
- Central: $P = \Gamma(E)$ the center of E:

$$J_p(a) = p \wedge a, \qquad p \in \Gamma(E).$$

- There always exist maximal compression bases.
- E can have different (maximal) compression bases with the same P.

Examples of compression bases

- Hilbert space effects $E(\mathcal{H})$: unique (maximal) compression base with
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- Hilbert space effects $E(\mathcal{H})$: unique (maximal) compression base with
$$P = P(\mathcal{H}),\ J_p(a) = pap.$$
- Effect algebras with RDP (MV-effect algebras): the central compression base is the unique (maximal) compression base.
- Direct product of compression bases: given a family of effect algebras E_i with compression bases $\{J_{i,p}\}_{p\in P_i}$, $i \in I$,
$$P = \Pi_{i\in I} P_i, \quad J_p = \Pi_{i\in I} J_{p_i},\ p = (p_i) \in \Pi_i P_i$$
is a compression base in $\Pi_{i\in I} E_i$.

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- The horizontal sum of Hilbert space effect algebras

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- Let φ be a faithful state on $E(\mathcal{H})$ ($\varphi(a) = 0$ implies $a = 0$).
- We construct a compression base with $P = P(\mathcal{H}) \hat{\oplus} P(\mathcal{H})$ and

$$\begin{aligned} J_{(p,0)}(a,0) &= (J_p(a),0), & J_{(p,0)}(0,a) &= (\varphi(a)p,0) \\ J_{(0,p)}(a,0) &= (0,\varphi(a)p), & J_{(0,p)}(0,a) &= (0,J_p(a)). \end{aligned}$$

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- we obtain many different compression bases with the same P.

Spectrality: projection cover property

$(E, \{J_p\}_{p\in P})$ - an effect algebra with a fixed compression base.

Definition (Gudder, 2006)

$(E, \{J_p\}_{p\in P})$ has the projection cover property if for any $a \in E$, there is a projection cover: $a^\circ \in P$ such that

$$a \leq p \iff a^\circ \leq p, \qquad \forall p \in P.$$

In this case, P is an OML.

Spectrality: b-property

Definition (SP, 2006)

$(E, \{J_p\}_{p \in P})$ has the b-property if for all $a \in E$, $q \in P$,

$$a \leftrightarrow q \iff P(a) \leftrightarrow q.$$

For $a, b \in E$, we say that a commutes with b (aCb) if $P(a) \leftrightarrow P(b)$.

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Lemma

Assume the b-property. Then for $p \in P$,

$$aCp \iff a \leftrightarrow p.$$

Spectrality: b-comparability

Definition (SP, 2006)

$(E, \{J_p\}_{p\in P})$ has the b-comparability property if

- it has the b-property
- for all $a, b \in E$, aCb, we have

$$\exists p \in P(a,b), \quad J_p(a) \le J_p(b),\ J_{p^\perp}(b) \le J_{p^\perp}(a).$$

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Lemma

Assume the b-comparability property. Then any $a \in E$ has a splitting projection:

$$p \in P(a):\ J_p(a) \leq J_p(1-a), \qquad J_{p^\perp}(a) \geq J_{p^\perp}(1-a)$$

$$\text{“}J_p(a) \leq 1/2\text{”} \qquad \text{“}J_{p^\perp}(a) \geq 1/2\text{”}$$

Spectrality

Definition (SP, 2006)

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Consequences (necessary conditions):

- $P = S(E) := \{p \in E,\ p \text{ is sharp}\}$ is an OML,
- E is covered by C-blocks

$$C(B) := \{a \in E, a \leftrightarrow B\} \quad \text{for a block } B \subseteq P$$

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  $\equiv$ maximal sets of mutually commuting elements
- any C-block is a spectral MV-effect algebra
- any $a \in E$ has a largest splitting projection

Binary spectral resolutions

Let $a \in E$. By repeated applications of splitting, we construct a family of projections

$$\{p_{a,\lambda(w)}\}_{w \in \{0,1\}^*},$$

indexed by binary fractions

$$\lambda(w) = \sum_{i=1}^{n} w_i 2^{-i}, \qquad w \in \{0,1\}^n$$

- the binary spectral resolution of a.

Characterization of the binary spectral resolution

Let E be archimedean and spectral. The binary spectral resolution $\{p_{\lambda(w)}\}_{w\in\{0,1\}^*}$ is the unique family in P

- $1 = p_1 \geq p_\lambda \geq p_\mu$ for $1 \geq \lambda \geq \mu$,
- $\bigwedge_{\lambda>\mu} p_\lambda = p_\mu$,
- $p_\lambda \leftrightarrow a$ for all λ,
- For n, $w \in \{0,1\}^n$, put

$$u_w := (p_{\lambda(w)+2^{-n}}) \wedge p'_{\lambda(w)} \in P,$$

then $f_w(J_{u_w}(a))$ exists in $[0, u_w]$, for the partially defined map

$$f_w = f_{w_n} \circ \cdots \circ f_{w_1}, \quad f_0(b) = 2b, \; f_1(b) = (2b^\perp)^\perp.$$

Spectral resolution in C-blocks

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Another way to construct a spectral resolution for $a \in E$:

- a is contained in a C-block C - a spectral MV-effect algebra
- C is the unit interval in a unital ℓ-group G
- C is spectral iff G is spectral $\implies$ there is the rational spectral resolution of a in G (Foulis):

$$\{p^C_{a,\lambda}\}_{\lambda \in \mathbb{Q} \cap [0,1]}$$

Spectral resolution in C-blocks

- this spectral resolution does not depend on the choice of C
- for binary fractions - the previous construction

$$p^C_{a,\lambda(w)} = p_{a,\lambda(w)}, \qquad w \in \{0,1\}^*.$$

- right continuity:

$$p^C_{a,\lambda} = \bigwedge_{\lambda(w)>\lambda} p_{a,\lambda(w)}, \qquad \lambda \in \mathbb{Q} \cap [0,1].$$

Dependence on the compression base

Theorem

Assume that E with $\{J_p\}_{p\in P}$ is spectral. Let $\{J'_p\}_{p\in P'}$ be another compression base with $P' = S(E)(= P)$. Then

- E with $\{J'_p\}_{p\in P'}$ is spectral
- the spectral resolutions are the same.

Further properties of spectral resolutions

If E has a separating family of states, then

- Any $a \in E$ is uniquely determined by its binary spectral resolution
- $a \in E$ is compatible with $q \in P$ if and only if $p_{a,\lambda(w)}$ is compatible with q for all binary fractions $\lambda(w)$.
- we have
$$``a = \int_{[0,1]} \lambda dp_{a,\lambda}"$$

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- An OMP is spectral $\implies$ it is a Boolean algebra.
- The horizontal sum $E(\mathcal{H})\hat{\oplus}E(\mathcal{H})$ is spectral.

Convex effect algebras

An effect algebra is convex if for every $a \in E$ and $\lambda \in [0,1]$ there is an element $\lambda a \in E$ such that

(C1) $\mu(\lambda a) = (\lambda\mu)a$.

(C2) If $\lambda + \mu \leq 1$ then $\lambda a \oplus \mu a \in E$ and $(\lambda + \mu)a = \lambda a \oplus \mu a$.

(C3) If $a \oplus b \in E$ then $\lambda a \oplus \lambda b \in E$ and $\lambda(a \oplus b) = \lambda a \oplus \lambda b$.

(C4) $1a = a$.

(Gudder et al., 1999)

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Theorem

Any convex effect algebra is affinely isomorphic to the interval $[0,u]$ in an ordered vector space (V, V^+) with an order unit u.

If (V, V^+, u) is an order unit space, we say that E is (strongly) archimedean.

(Gudder et al., 1999)

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- If E is strongly archimedean, then any compression is also affine and extends to a positive linear operator on V.
- A convex MV-effect algebra is spectral $\iff$ it is monotone σ-complete.
- The notion of spectral duality for order unit spaces (Alfsen and Schultz, 1976, 2003) is strictly stronger than Foulis spectrality .

Example: Generalized spin factors

A generalized spin factor is an order unit space defined from a Banach space $(X, \|\cdot\|)$ (Berdikulov and Odilov, 1994):

$$V = \mathbb{R} \times X, \quad V^+ = \{(\alpha, x),\ \|x\| \leq \alpha\}, \quad u = (1, 0)$$

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Let $(X, \|\cdot\|)$ be reflexive. Then

- V is Foulis spectral $\iff$ $(X, \|\cdot\|)$ is strictly convex.
- A is is spectral duality $\iff$ $(X, \|\cdot\|)$ is strictly convex and smooth.

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- True for
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- Question: E is spectral $\overset{?}{\Longleftrightarrow}$ G is spectral (in Foulis sense)?
- True for
  - MV-effect algebras
  - archimedean divisible (convex) effect algebras
- False in general.

  Counterexample: the horizontal sum $E(\mathcal{H}) \hat{\oplus} E(\mathcal{H})$

  (an interval effect algebra which is spectral but its universal group is not unperforated hence not spectral).