Monadic adjunctions for quantum logics

Gejza Jenča

Slovak University of Technology in Bratislava

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The plan

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- Effect algebras and other quantum logics

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- Effect algebras are algebras for the Kalmbach monad on **BPos**

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- Other monadic adjunctions:
    - Pseudo-effect algebras over **BPos**

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  - ω-effect algebras over **BPos**
  - Orthomodular posets over **BPosInv**

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  - ω-effect algebras over **BPos**
  - Orthomodular posets over **BPosInv**
- Future research

Effect Algebras

Foulis and Bennett 1994; Kôpka and Chovanec 1994; Giuntini and Greuling 1989

An effect algebra is a partial algebra $(E; +, 0, 1)$ satisfying the following conditions.

(E1) If $a + b$ is defined, then $b + a$ is defined and $a + b = b + a$.

(E2) If $a + b$ and $(a + b) + c$ are defined, then $b + c$ and $a + (b + c)$ are defined and $(a + b) + c = a + (b + c)$.

(E3) For every $a \in E$ there is a unique $a' \in E$ such that $a + a' = 1$.

(E4) If $a + 1$ exists, then $a = 0$

Basic Relationships

Let E be an effect algebra.

- Cancellativity: $a + b = a + c \Rightarrow b = c$.
- Partial difference: If $a + b = c$ then we write $a = c - b$. The operation $-$ is well defined and $a' = 1 - a$.
- Poset: Write $b \leq c$ iff $\exists a : a + b = c$; $(E, \leq)$ is then a bounded poset.

Classes of Effect Algebras

The class of effect algebras includes

- modular ortholattices (Birkhoff and Von Neumann, 1936)
- orthomodular lattices (Husimi, 1937)
- orthomodular posets (Finch, 1970)
- orthoalgebras (Foulis and Randall, 1981)
- MV-algebras (Chang, 1959)
- Any interval $[0, u]$ in the positive cone of an abelian po-group.
- Boolean algebras.

Orthomodular posets
(Finch, 1970)

An orthomodular poset is a bounded poset with involution $(A, \leq, ', 0, 1)$ satisfying the following conditions, for all $x, y \in A$.

- $x \wedge x' = 0$.
- If $x \leq y'$, then $x \vee y$ exists.
- If $x \leq y$, then $x \vee (x \vee y')' = y$.

An orthomodular lattice is an orthomodular poset that is a lattice.

The Kalmbach construction

(Kalmbach, 1977; Mayet and Navara, 1995)

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- Let $K(A)$ be the set of all finite chains in A with even number of elements.
- Introduce a partial order on the set $K(A)$ by the following rule:

$$[a_1 < a_2 < \cdots < a_{2n-1} < a_{2n}] \leq [b_1 < b_2 < \cdots < b_{2n-1} < b_{2k}]$$

if and only if for every $i \in \{1, \ldots, n\}$ there exists $j \in \{1, \ldots, k\}$ such that $b_{2j-1} \leq a_{2i-1} < a_{2i} \leq b_{2j}$.

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- Equip $K(A)$ with the unary operation $C \mapsto C'$ given by the rule

$$C' = C \Delta \{0, 1\}$$

  where Δ is the symmetric difference.

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Then $(K(A), \leq, ')$ is an orthomodular poset and $\eta_A : A \to K(A)$ given by

$$\eta_A(a) = \begin{cases} [0 < a] & \text{if } 0 < a \\ \emptyset & \text{if } a = 0 \end{cases}$$

is an embedding of A into $K(A)$.

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Moreover, if A is a lattice, then $K(A)$ is an orthomodular lattice and η_A is a bounded lattice homomorphism. (This is the original Kalmbach's result).

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Corollary

Every bounded lattice is a bounded sublattice of an orthomodular lattice.

Where does the Kalmbach construction come from

Theorem

Harding (2004) K is a functor left adjoint to the forgetful functor G from the category of orthomodular posets to the category of bounded posets.

$$
\begin{tikzcd}
\mathbf{BPos} \arrow[rr, bend left, "K"] \arrow[rr, phantom, "\perp"] & & \mathbf{OMP} \arrow[ll, bend left, "G"]
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However, K does not restrict to a functor from bounded lattices (with lattice homomorphisms) to orthomodular lattices.

Adjunctions

$$\begin{tikzcd} \mathbf{BPos} \arrow[rr, bend left, "K"] \arrow[rr, phantom, "\perp"] & & \mathbf{OMP} \arrow[ll, bend left, "G"] \end{tikzcd}$$

For every bounded poset A and every OMP B there is a bijection between hom-sets

$$\mathrm{Hom}_{\mathbf{BPos}}(A, G(B)) \simeq \mathrm{Hom}_{\mathbf{OMP}}(K(A), B)$$

Adjunctions

$$\begin{tikzcd} \mathbf{BPos} \arrow[r, bend left, "K"] \arrow[r, phantom, "\perp"] & \mathbf{OMP} \arrow[l, bend left, "G"] \end{tikzcd}$$

- Take an arbitrary bounded poset A.
- Take an OMP B and morphism of bounded posets $g\colon A \to G(B)$

There is a unique morphism of orthomodular posets $f\colon K(A) \to B$ such that

$$\begin{tikzcd} GK(A) \arrow[r, "G(f)"] & G(B) \\ A \arrow[u, "\eta_A"] \arrow[ur, "g"] & \end{tikzcd}$$

commutes.

Adjunctions and monads

- Every adjunction induces a monad on the domain category of the left adjoint functor.
- Every monad T on a category $\mathcal{C}$ gives rise to a category of algebras $\mathcal{C}^T$ and an adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[rr, bend left] \arrow[rr, phantom, "\perp"] & & \mathcal{C}^T \arrow[ll, bend left] \end{tikzcd}$$

- For every adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[rr, bend left] \arrow[rr, phantom, "\perp"] & & \mathcal{D} \arrow[ll, bend left] \end{tikzcd}$$

  that induces T there is a canonical comparison functor $\mathcal{D} \to \mathcal{C}^T$.
- An adjunction is <u>monadic</u> if the comparison functor is an equivalence of categories.

Is this adjunction monadic?

$$\begin{tikzcd} \mathbf{BPos} \arrow[rr, bend left, "K"] \arrow[rr, phantom, "\perp"] & & \mathbf{OMP} \arrow[ll, bend left, "U"] \end{tikzcd}$$

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Answer

No.

The category of algebras for the Kalmbach monad

We say that the monad on **BPos** induced by the adjunction between **BPos** and **OMP** is the Kalmbach monad

Theorem

(GJ, 2015) The category of algebras for the Kalmbach monad is equivalent to the category of effect algebras **EA**.

Pseudo effect algebras

Definition

Dvurečenskij and Vetterlein (2001) A pseudo effect algebra is an algebra A with a partial binary operation $+$ and two constants $0, 1$ such that, for all $a, b, c \in A$.

- (PE1) If $a + (b + c)$ exists, then $(a + b) + c$ exists and $a + (b + c) = (a + b) + c$.
- (PE2) There is exactly one d and exactly one e such that $a + d = e + a = 1$.
- (PE3) If $a + b$ exists, there are d, e such that $d + a = b + e = a + b$.
- (PE4) If $a + 1$ exists or $1 + a$ exists, then $a = 0$.

Pseudo effect algebras are algebras for a monad on **BPos**

Theorem

(GJ, 2020) The forgetful functor from the category of pseudo effect algebras to the category of bounded posets is a right adjoint functor of a monadic adjunction.

ω-effect algebras

Definition

We say that an effect algebra E is ω-effect algebra when every increasing sequence $a_1 \leq a_2 \leq \cdots$ in E has a supremum. A morphism of ω-effect algebras is a morphism of effect algebras that preserves suprema of increasing sequences.

ω-effect algebras are algebras for a monad on **BPos**

Theorem

(van de Wetering, 2021) The forgetful functor from the category ω-effect algebras to the category of bounded posets is a right adjoint functor of a monadic adjunction.

What about orthomodular posets?

- Recall, that there is an adjunction between **BPos** and **OMP**.
- However, this adjunction is non-monadic.
- Can we represent orthomodular posets as algebras for a monad?

Orthomodular posets are algebras for a monad on **BPosInv**

Theorem

(GJ, 2022) The forgetful functor from the category **OMP** *to the category of bounded posets with involution is a right adjoint functor of a monadic adjunction.*

Conclusion, future research

There are monadic adjunctions

$$\begin{tikzcd} \mathbf{BPos} \arrow[rr, bend left] \arrow[rr, phantom, "\perp"] & & \mathbf{PsEA} \arrow[ll, bend left] \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{BPos} \arrow[rr, bend left] \arrow[rr, phantom, "\perp"] & & \omega\mathbf{EA} \arrow[ll, bend left] \end{tikzcd}$$

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The proof of all these uses

- General adjoint functor theorem
- Beck's monadicity theorem

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Problem

Give an explicit description of the left adjoint functor in these adjunctions.

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