THE MÖBIUS FUNCTION ON A POSET

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Though the analysis here is self-contained, we give a reference: Chapter 3 of [Stan].

1. Setup

For this section, we fix a commutative ring A with unity, an A-(bi)module M (since A is commutative), and a partially ordered set $(X, \leq)$ that is *locally finite* (this is defined presently). In order to define (and prove stuff about) the Möbius function on X, we first look at a set of functions with a group structure of convolution on it, just like in the classical case of arithmetic functions on natural numbers.

Definitions. Suppose $(X, \leq)$ is a partially ordered set.

(1) We say X is *locally finite* if for all $x \leq y$ in X, the interval $[x, y] := \{z \in X : x \leq z \leq y\}$ is finite.
(2) Define $I \subset X \times X$ to be the set of pairs (x, y) so that $x \leq y$.
(3) Now define $\mathcal{B}$ to be the set of functions $f : I \to A$ (if desired, they can be extended to $f : X \times X \to A$ by setting $f(x, y) = 0$ if $(x, y) \notin I$).
   Also define $\mathcal{M}$ to be the set of functions from I to M.
(4) Say X is locally finite. We then define the *convolution operation* $* : \mathcal{B} \times \mathcal{M} \to \mathcal{M}$ sending $(f, g) \in \mathcal{B} \times \mathcal{M}$ to $f * g = f \cdot g$, by

$$(f * g)(x, y) = \sum_{z \in [x,y]} f(x, z)g(z, y)$$

   (Note that each such sum is only over finitely many terms.) We can similarly define the convolution operation $* : \mathcal{M} \times \mathcal{B} \to \mathcal{M}$.
(5) We will also need to consider the subclass $\mathcal{I}$ of functions $f \in \mathcal{B}$ such that $f(x, x) \in A^\times$ for all $x \in X$.

A special case of such an operation is when we take $M = A$, and $\mathcal{M} = \mathcal{B}$. This gives $* : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, and we presently show that under this operation, $\mathcal{B}$ is a monoid.

Example. The Möbius function (among all arithmetic functions) is one such example, where we have $X = \mathbb{N}$ and $A = M = \mathbb{Z}$. The partial order on $\mathbb{N}$ is the order $x \leq y$ iff $x|y$. This satisfies all the conditions above, and given $x \leq y$ we define $f \in \mathcal{B}$ from $\mathbb{N}$ to $\mathbb{Z}$ by $f(x, y) := f(y/x)$. Then the convolution operation is the standard one:

$$(f * g)(x, y) = \sum_{z \in [x,y]} f(x, z)g(z, y) = \sum_{(z/x)|(y/x)} f(z/x)g(y/z) = \sum_{d|n} f(d)g(n/d)$$

where $n = y/x$ and $d = (z/x)|n$ (so that $n/d = y/z$). The analysis of the Möbius function done below, now specializes to exactly the classical analysis of the Möbius function for $\mathbb{N}$.

2. The main result

Theorem.

1. *$(\mathcal{B}, +, *)$ is a ring, whose group of units (i.e. $*$-invertible elements) is $\mathcal{B}^\times = \mathcal{I}$.*
2. *$\mathcal{M}$ is a left- and a right- module over $\mathcal{B}$.*

Proof. We first observe that $\mathcal{B}, \mathcal{M}$ are A-modules, under the obvious addition (pointwise over I) and scalar left-multiplication by $a \in A$. It is also easy to check that $*$ distributes over $+$.

Next, we show that $*$ is associative. Given $f, g \in \mathcal{B}$ and $m \in \mathcal{M}$, we compute for $x \leq y$:

$$\begin{aligned}((f * g) \cdot h)(x, y) &= \sum_{x \leq z \leq y} (f * g)(x, z)h(z, y) = \sum_{x \leq w \leq z \leq y} f(x, w)g(w, z)h(z, y) \\ (f \cdot (g \cdot h))(x, y) &= \sum_{x \leq w \leq y} f(x, w)(g * h)(w, y) = \sum_{x \leq w \leq z \leq y} f(x, w)g(w, z)h(z, y)\end{aligned}$$

Setting $M = A$, we get associativity for $*$ in $\mathcal{B}$.

We now claim that $\mathcal{B}$ has a two-sided identity under $*$. This is shown by defining $e(x, y) = \delta_{x,y}$. We now have (for $x \leq y$):

$$(f * e)(x, y) = \sum_{x \leq z \leq y} f(x, z)e(z, y) = \sum_{x \leq z \leq y} f(x, z)\delta_{z,y} = f(x, y)\delta_{y,y} = f(x, y)$$

and similarly we show that $(e * f) = f$, whence e is the two-sided identity in $\mathcal{B}$. The proof that $e \cdot h = h$ for all $h \in \mathcal{M}$ is similar too. This completes the proof that $\mathcal{B}$ is a ring, as well as the fact that $\mathcal{M}$ is a left-module over $\mathcal{B}$. The proof that $\mathcal{M}$ is a right-module is similar. Note, though, that $\mathcal{B}$ is not commutative in general (this depends on the poset structure of X), and hence $\mathcal{M}$ is not a $\mathcal{B}$-bimodule.

Before showing that $\mathcal{I}$ is precisely the set of invertible functions (or units) in $\mathcal{B}$, let us remark that if any f in a monoid $(\mathcal{B}, *, e)$ has a left or a right inverse, namely $g_L = g$ or $g_R =$ respectively, then both inverses exist, and $g_L = g_R = g$. This is standard, because by associativity of $*$, we have

$$g_L = g_L * e = g_L * (f * g) = (g_L * f) * g = e * g = g = \cdots = g_R$$

Finally, we consider invertible elements in $\mathcal{B}$. We first claim that if $f \in \mathcal{B}$ is invertible, then $f(x, x) \in A^\times$. To see this, if $g = f^{-1}$, then evaluating $f * g = e$ at (x, x) for any x, we have

$$1 = e(x, x) = (f * g)(x, x) = \sum_{x \leq z \leq x} f(x, z)g(z, x) = f(x, x)g(x, x)$$

(The proof for $g * f = e$ is the same.)

The converse is harder to show. Suppose $f(x, x) \in A^\times$ for all $x \in X$. We now inductively define a *right-inverse* g_R to f at (x, y) for all $x \leq y$, where we apply induction on $|[x, y]|$ (i.e. the size of the interval $[x, y]$, or the number of elements z so that $x \leq z \leq y$). For $|[x, y]| = 1$, the only possibility is when $x = y$, and we define $g_R(y, y) = f(y, y)^{-1} \in A^\times$.

Now suppose that we have defined $g_R(x, y)$ for all $x \leq y$, where $|[x, y]| < n$ for some $n > 0$. Consider any $x \leq y$ such that $|[x, y]| = n$. We then define

$$g_R(x, y) := f(x, x)^{-1}[\delta_{x,y} - \sum_{x < z \leq y} f(x, z)g_R(z, y)]$$

(Note that this implies that

$$f(x,x)g_R(x,y) + \sum_{x<z\leq y} f(x,z)g_R(z,y) = \delta_{x,y}$$

i.e. $(f * g_R)(x,y) = e(x,y)$ as desired.)

Moreover, the above definition makes sense, since each summand on the right side is already defined, since $|[z,y]| < |[x,y]| = n$ (this is because $x < z$, so $[z,y] \subset [x,y]$, but $x \notin [z,y]$). We also observe that this definition is forced upon us by the equation $f * g_R = e$.

The proof for the existence of a left-inverse g_L is similar. Thus both inverses exist iff $f \in \mathcal{I}$, and by the above remarks they must coincide. Hence $\mathcal{I}$ is indeed the group of units in $\mathcal{B}$ (it is now standard to show that the inverse is unique etc.)

□

3. Möbius inversion formulae

We next show the Möbius inversion formula - or two versions of it (the first version is stated in a "left" as well as a "right" way).

Proposition 1. *Henceforth, let M merely denote an abelian group.*

(1) *There exists a unique function $\mu : X \times X \to A$, called the* Möbius function*, so that $\mu(x,y) = 0$ unless $x \leq y$, and*

$$\sum_{x\leq z\leq y} \mu(x,z) = \delta_{x,y}$$

Moreover, μ actually has values in $\mathbb{Z}$ (or its image in A), and also satisfies the "dual" identity, namely:

$$\sum_{x\leq z\leq y} \mu(z,y) = \delta_{x,y} \ \forall x \leq y$$

(2) (Möbius inversion formula 1.) *If $f : I \to M$, define $h_L(x,y) := \sum_{x\leq z\leq y} f(z,y)$ and $h_R(x,y) := \sum_{x\leq z\leq y} f(x,z)$. Then*

$$f(x,y) = \sum_{x\leq z\leq y} \mu(x,z)h_L(z,y) = \sum_{x\leq z\leq y} \mu(z,y)h_R(x,z)$$

(3) (Möbius inversion formula 2.) *Suppose for each $x \in X$, that the set $\{y \in X : y \leq x\}$ is finite. If $F : X \to M$, define $H_R(x) := \sum_{y\leq x} F(y)$. Then*

$$F(x) = \sum_{z\leq x} \mu(z,x)H_R(z)$$

We show another example of Möbius inversion below, after the proof.

Proof. Firstly, note for the two inversion formulas, that the expression makes sense since μ takes values in $\mathbb{Z}$ by the first part. Moreover, we can write μ to the left or right since M is a $\mathbb{Z}$-bimodule (since $\mathbb{Z}$ is commutative).

Next, let us define the function $U : I \to A$ by $U \equiv 1$. Thus $U \in \mathcal{I}$.

(1) The two desired identities are merely saying that $\mu * U = U * \mu = e$ in $\mathcal{B}$. This unique two-sided inverse to U under $*$ in $\mathcal{B}$ now exists by the previous theorem. Moreover, since $U \in \mathcal{B}_{\mathbb{Z}} := \{f : X \to \mathbb{Z}\}$, hence we also have $\mu = U^{-1} \in \mathcal{B}_{\mathbb{Z}}$. Note here that $\mathcal{B}_{\mathbb{Z}} \subset \mathcal{B}$ since we have $\varphi : \mathbb{Z} \to A$, sending $1 \mapsto 1$, which sends $f : X \to \mathbb{Z}$ to $\varphi \circ f : X \to \mathbb{Z} \to A$.

(2) This assertion is also clear, since we clearly have $h_L = U * f$ in the left $\mathcal{B}_{\mathbb{Z}}$-module $\mathcal{M}$, and $h_R = f * U$ in the right $\mathcal{B}_{\mathbb{Z}}$-module $\mathcal{M}$. By the module structure, we thus have

$$f = e * f = (\mu * U) * f = \mu * (U * f) = \mu * h_L$$

which is exactly what is claimed. The proof that $f = h_R * \mu$ is similar.

(3) One way to verify this is to use directly compute, noting that each sum is finite by our assumption on X:

$$\begin{aligned}\sum_{z \leq x} \mu(z,x) H_R(z) &= \sum_{z \leq x} \mu(z,x) \sum_{y \leq z} F(y) = \sum_{y \leq z \leq x} F(y)\mu(z,x) = \sum_{y \leq x} F(y) \sum_{z \in [y,x]} \mu(z,x) \\ &= \sum_{y \leq x} F(y)\delta_{y,x} = F(x)\end{aligned}$$

where we use the first part of the identity (or perhaps the dual of it) for one of the steps.

The other way to verify these formulae are to use a slightly different poset, and the verified module structure and Möbius function on that poset.

We attach a *least* element 0 to X, to get another poset $X' = X \cup \{0\}$ with $0 < x \ \forall x \in X$. Note then that we can extend U to $U' \equiv 1$ on X', and the function μ on X also extends to μ'. In other words, the inverse of U' in $\mathcal{B}_{\mathbb{Z},X'}$ restricts to μ on X - this follows from the uniqueness property of μ.

We now define $f : I_{X'} \to \mathbb{Z}$ by $f(0,x) = F(x)$ for all $x \in X$, and any arbitrary values for the others (as we shall see, the only value that might matter is that of $f(0,0)$, but even this does not matter !). We also define $H_R(0) = F(0) := f(0,0)$. For $x \in X$, we then have

$$\begin{aligned}H_R(x) &= \sum_{y \leq x} F(y) = \sum_{0<y\leq x} f(0,y) = \sum_{0 \leq y \leq x} f(0,y)U(y,x) - f(0,0)U(0,x) \\ &= (f * U)(0,x) - f(0,0)\end{aligned}$$

For $x = 0$, we also observe that

$$(f * U)(0,0) = f(0,0)U(0,0) = f(0,0) = F(0) = H_R(0)$$

if we extend U to $U' \equiv 1$ on X'. Using these equations, we now compute the desired expression:

$$\begin{aligned}
\sum_{z \le x} \mu(z,x) H_R(z) &= \sum_{0 < z \le x} \mu(z,x) H_R(z) \\
&= \sum_{0 < z \le x} \mu(z,x)[(f * U)(0,z) - f(0,0)] + \mu(0,x) H_R(0) - \mu(0,x) H_R(0) \\
&= \sum_{0 \le z \le x} \mu(z,x)(f * U)(0,z) - f(0,0) \sum_{0 < z \le x} \mu(z,x) - \mu(0,x) f(0,0) \\
&= ((f * U) * \mu)(0,x) - f(0,0) \sum_{0 \le z \le x} \mu(z,x) \\
&= f(0,x) - f(0,0) \sum_{0 \le z \le x} U'(0,z)\mu'(z,x) \\
&= F(x) - (U' * \mu')(0,x) = F(x) - \delta_{0,x} = F(x)
\end{aligned}$$

since $x \in X$. Hence we are done. (Also observe that the proof is independent of the specific other values chosen for f at various points in $I \subset X \times X$.)

Note also, that if X has the property that for any $x \in X$, the set $R_x := \{y \in X : y \ge x\}$ is finite, then one can define H_L and carry out a similar analysis for the "other-handed" case here. To show this left-handed version, we work instead with a different poset $X'' := X \cup \{\infty\}$, with $x < \infty \ \forall x \in X$. The equations and proof are similar.

□

4. Some easy results

We now have the following corollary to the Möbius inversion formula:

Corollary 1. *For all $x \le y \in X$, we have*

$$\sum_{x \le z \le y} \mu(x,z)|[z,y]| = \sum_{x \le z \le y} \mu(z,y)|[x,z]| = 1$$

where $|[x,y]|$ is the size of that interval in X (and finite by assumption).

Proof. Let us evaluate $(U * U)$ at any point of I. We have

$$(U * U)(x,y) = \sum_{x \le z \le y} U(x,z)U(z,y) = \sum_{x \le z \le y} 1 = |[x,y]|$$

and therefore the claimed result just says that $(\mu * (U * U))(x,y) = 1 = U(x,y)$, and that $((U * U) * \mu)(x,y) = 1 = U(x,y)$. This follows from Möbius inversion, as above. □

We next compute the Möbius function over small posets.

Proposition 2. *If $x, y, z \in X$, with $[x,y] = \{x,y\}$ and $[x,z] = \{x,y,z\}$, then $\mu(x,x) = 1$, $\mu(x,y) = -1$, and $\mu(x,z) = 0$.*

Proof. This is trivial, if we just compute that $(U*\mu)(x,x) = 1$, $(U*\mu)(x,y) = (U*\mu)(x,z) = 0$, and expand these out. □

Finally, we show an easy result (that applies to the example $X = \mathbb{Z}$, among others) that implies the commutativity of $\mathcal{B}$.

Lemma. *Suppose for each $x \leq y$ in X, we have a permutation $\sigma_{x,y}$ of the finite set $[x, y]$, that interchanges x and y. Now define I' to be the quotient of $I = \{(x, y) \in X \times X : x \leq y\}$ by the relations $\{(x, z) = (\sigma_{x,y}(z), y)$ for all $x \leq y \leq z \in X\}$, and suppose $f, g : I' \to A$. Then $f * g = g * f$.*

As an example, consider $X = \mathbb{Z}$. We know that $f(x, y) = f(y/x)$, and we define $\sigma_{x,y}(z) = xy/z$ for all $x|z|y$. Then we verify that $\sigma^2_{x,y}(z) = z$ for all $z \in [x, y]$. Moreover, the relation says that

$$f(z/x) = f(x, z) = f(\sigma_{x,y}(z), y) = f(y/\sigma_{x,y}(z)) = f(y/[xy/z]) = f(z/x)$$

as it should.

Proof. This is easy: we use the fact that summing over $z \in [x, y]$ is the same as summing over $\sigma_{x,y}(z)$, by the given assumptions. Hence we compute, for general $x \leq y \in X$, using the given properties:

$$\begin{aligned}(f * g)(x, y) &= \sum_{z \in [x,y]} f(x, z)g(z, y) = \sum_{z \in [x,y]} f(\sigma_{x,y}(z), y)g(x, \sigma_{x,y}(z)) \\ &= \sum_{\sigma_{x,y}(z) \in [x,y]} g(x, \sigma_{x,y}(z))f(\sigma_{x,y}(z), y) = (g * f)(x, y)\end{aligned}$$

and since this holds for all $x \leq y$, we are done. □

5. Examples

Example 1: The classical Möbius function. (We prove this result below, using results on functoriality, and the next example.) Let $(X, \leq)$ be the set $\mathbb{N}$ with the partial order of divisibility. Then it is well-known that the Möbius function here (for any $d, n \in \mathbb{N}$) is

$$\mu_{\mathbb{N}}(n) = \mu(d, dn) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^r & \text{if for some distinct primes } p_1, \ldots, p_r,\ n = p_1 \ldots p_r \\ 0 & \text{otherwise} \end{cases}$$

Example 2: Another poset structure for the natural numbers. We now endow $\mathbb{N}$ with the usual partial - or total, in this case - order inherited from $\mathbb{R}$. We now present its Möbius function:

Proposition 3. *For $m \leq n$, the Möbius function is*

$$\mu(m, n) = \begin{cases} 1 & \textit{if } n - m = 0 \\ -1 & \textit{if } n - m = 1 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. This will follow from the results on functoriality that we present below, but here is the proof anyways. Firstly, $\mu(m,m)=1$ and $\mu(m,m+1)=-1$ for all m, by a proposition above. Next, we claim by induction that $\mu(m,m+1+n)=0$ for all $n\in\mathbb{N}$. The base case of $n=1$ also follows from the proposition above. The poset structure gives us merely that

$$\begin{aligned} 0 &= (U*\mu)(m,m+1+n) = \mu(m,m)+\mu(m,m+1)+\sum_{j=0}^{n-1}\mu(m,m+2+j) \\ &= \sum_{j=0}^{n-2}\mu(m,m+2+j)+\mu(m,m+n+1) = \mu(m,m+n+1) \end{aligned}$$

□

Example 3: Finite subsets of a set. For any set S, its power set $\mathcal{P}(S)$ is a poset, with inclusion as the partial order. If we look at the set of finite subsets of S, then this is clearly an interval-finite poset. (This equals the entire power set if S is finite.) Let us determine the Möbius function of this poset.

Proposition 4. *For $V\subset W\subset S$ with W finite, the Möbius function is $\mu(V,W)=(-1)^{|V|+|W|}$.*

Proof. The proof is by induction on $n=|W|-|V|$. For $n=0$, we have $V\subset W$ and $|V|=|W|$, hence $V=W$. But then $\mu(V,V)=\mu(V,V)U(V,V)=(\mu*U)(V,V)=1$, as desired. Now suppose we know the result for all $n<K$, and let $|W|-|V|=K$. Then we have

$$\sum_{Z\in[V,W]}\mu(V,Z)=0$$

so we get that

$$\mu(V,W)=-\sum_{Z\in[V,W)}\mu(V,Z)$$

Now note that if $W=V\coprod\{s_1,\dots,s_K\}$ (where $s_i\in S$), then the subsets $Z\in[V,W]$ are characterized exactly by the s_i's that are contained in Z. Thus for all $0\le j\le K$, there are exactly $\binom{K}{j}$ subsets Z of W, that contain exactly j of the s_i's. And for each of these Z's, we have $\mu(V,Z)=(-1)^j$, by the induction hypothesis. In particular, we have

$$\mu(V,W)=-\sum_{V\le Z<W}\mu(V,Z)=-\sum_{j=0}^{K-1}\binom{K}{j}(-1)^j=-(1-1)^K+(-1)^K=(-1)^K$$

and hence we are done, since $(-1)^K=(-1)^{|W|-|V|}=(-1)^{|W|+|V|}$. □

Example 4: The Bruhat order. Let $X=W$ be any Coxeter group, with $\le$ the Bruhat order on it. It is stated in [Hum], that $\mu(x,z)=(-1)^{l(x)+l(z)}$ for all $x\le z$ in W.

Example 5: Möbius functions with any integer value. We could ask the question, given the above examples: Does the Möbius function, which is integer-valued, only take on the values 0 and ±1?

The answer is no: let us construct a two-parameter family of posets $X_{m,n}$, each with unique extremal elements x,y, with various values of $\mu(x,y)$.

Given $m, n \geq 0$, define a poset structure on the set

$$X_{m,n} := \{x, w_1, w_2, \dots, w_{m+1}, z_1, \dots, z_{n+1}, y\}$$

by: $x < w_j < z_i < y \ \forall i, j$.

We now compute the various μ-values. Firstly, $\mu(x,x) = \mu(w_j), w_j) = \mu(z_i, z_i) = \mu(y,y) = 1$ and $\mu(x, w_j) = \mu(w_j, z_i) = \mu(z_i, y) = -1$, by a proposition above, for all i, j. Next, we have

$$\mu(x, z_i) = -\mu(x,x) - \sum_j \mu(x, w_j) = -1 - (m+1)(-1) = m$$

for all i. Similarly, for each j, we have

$$\mu(w_j, y) = -\mu(w_j, w_j) - \sum_i \mu(w_j, z_i) = -1 - (n+1)(-1) = n$$

Finally, we compute $\mu(x, y)$. This equals

$$\begin{aligned} -\mu(x,x) - \sum_j \mu(x, w_j) - \sum_i \mu(x, z_i) &= -1 - (m+1)(-1) - (n+1)m \\ &= -1 + m + 1 - (n+1)m = -mn \end{aligned}$$

Thus, the Möbius function can assume all possible integer values.

6. Functoriality

We now relate the Möbius functions in several different setups.

Proposition 5.

(1) *If $\varphi : X \to Y$ is an isomorphism of locally finite posets, then for all $x, x' \in X$, we have: $\mu_X(x, x') = \mu_Y(\varphi(x), \varphi(x'))$.*

(2) *If $X \subset Y$ is "interval-closed" (i.e. if $x, y \in X$ then $[x, y]_Y \subset X$), and Y is locally finite, then $\mu_X = \mu_Y|_X$.*

(3) *If X_i are locally finite posets, then the disjoint union $X = \coprod_i X_i$ is also a locally finite poset, with Möbius function equal to*

$$\mu(x_i, x_j) = \delta_{ij} \mu_{X_i}(x_i, x_j)$$

where $x_i \in X_i, x_j \in X_j$.

(4) *If X_i are (finitely many) locally finite posets, then their product $X = \prod_i X_i$ (together with the partial order $(x_i)_i \leq (y_i)_i$ iff $x_i \leq_i y_i \ \forall i$) has Möbius function*

$$\mu((x_i)_i, (y_i)_i) = \prod_i \mu_i(x_i, y_i)$$

Proof. (1) This is obvious - and is also a consequence of the next part!

(2) We need to show that $\mu_X(x, y) = \mu_Y(x, y)$ for $x, y \in X$. Note that $[x, y]_X = [x, y]_Y$, whence it is easy to see that $\mu_X(x, y) = \mu_{[x,y]}(x, y) = \mu_Y(x, y)$.

(3) This is also easy to see, since the intervals are of the form $[x_i, x'_i] = [x_i, x'_i]_{X_i}$ for all i and all $x_i, x'_i \in X_i$.

(4) We claim, firstly, that each interval is the product of the respective intervals. In other words, given $x_i \leq_i y_i$ in X_i for all i, we claim that $[(x_i)_i, (y_i)_i]_X = \prod_i [x_i, y_i]_{X_i}$. This is because we have

$$(x_i)_i \leq (z_i)_i \leq (y_i)_i \iff x_i \leq z_i \leq y_i \ \forall i$$

We now claim that $\prod_i \mu_i$ is a (and hence "the") Möbius function on X, for we compute that $\mu((x_i)_i, (x_i)_i) = \prod_i \mu_i(x_i, x_i) = \prod_i 1 = 1$, and for $(x_i)_i < (y_i)_i$, there is a j so that $x_j < z_j$, so

$$\begin{aligned}\sum_{(x_i)_i \leq (z_i)_i \leq (y_i)_i} \mu((x_i)_i, (z_i)_i) &= \sum_{z_i \in [x_i, y_i] \ \forall i} \prod_j \mu_j(x_j, z_j) = \prod_i \Big(\sum_{z_i \in [x_i, y_i]} \mu_i(x_i, z_i) \Big) \\ &= \prod_i (\mu_i * U_i)(x_i, z_i) = \prod_i \delta_{x_i, z_i} = 0\end{aligned}$$

□

We now define the *lower one-point compactification* $X_{-\infty}$ of a poset X. Define $X_{-\infty}$ to be the set $X \cup \{-\infty\}$, with the relation that $-\infty < x \ \forall x \in X$.

Proposition 6. *Suppose X, Y are posets, with Y finite, and X locally finite. Let us define a new poset Z by "superimposing" X after Y. In other words, $y < x$ for all $y \in Y, x \in X$. Then the Möbius functions are related as follows: define, for each $y \in Y$, the integer $n_y = \sum_{y' \geq y} \mu_Y(y, y')$. Then for all $y, y' \in Y, x, x' \in X$, we have*

$$\mu_Z(y, y') = \mu_Y(y, y'), \ \mu_Z(x, x') = \mu_Z(x, x'), \ \mu_Z(y, x) = n_y \mu_{X_{-\infty}}(-\infty, x)$$

Proof. The first two assertions follow from one of the parts of the previous proposition, so it remains to show the last part. Let us now prove that $\sum_{z \in [y,x]} \mu_Z(y, z) = 0$; this completes the proof. We observe that

$$\begin{aligned}\sum_{z \in [y,x]} \mu_Z(y, z) &= \sum_{z \in Y \cap [y,x]} \mu_Z(y, z) + \sum_{z \in X \cap [y,x]} \mu_Z(y, z) = n_y + \sum_{z \in X \cap [y,z]} n_y \mu_{X_{-\infty}}(-\infty, z) \\ &= n_y \mu_{X_{-\infty}}(-\infty, -\infty) + n_y \sum_{z \in X \cap [y,z]} \mu_{X_{-\infty}}(-\infty, z) \\ &= n_y \sum_{z \in X_{-\infty} \cap [-\infty, x]} \mu_{X_{-\infty}}(-\infty, z) = n_y (U * \mu_{X_{-\infty}})(-\infty, x) = 0\end{aligned}$$

□

Corollary 2. *In the same setup, suppose Y has a unique maximum element y_{max}. Then $n_y = 0$ for all $y \neq y_{max}$, whence $\mu(y, x) = 0$ for all $x \in X, y \in Y \setminus \{y_{max}\}$.*

A straightforward application is for $Y = \{n, n+1\}$ and $X = [n+2, \infty) \cap \mathbb{N}$, under the partial order $m \leq n$ if $n - m \geq 0$. Then we get immediately that $\mu(n, n+1+m) = 0$ for all $m \in \mathbb{N}$, in the setup of Example 2.

Proof. This is because $n_y = \sum_{y \in [y, y_{max}]} \mu(y, z) = (U * \mu_Y)(y, y_{max}) = \delta_{y, y_{max}} = 0$. □

We conclude by computing the classical Möbius function on $\mathbb{N}$.

Corollary 3. *The classical Möbius function for* $(\mathbb{N}, \cdot|\cdot)$ *is* $(-1)^r$ *at a product of any number* $r \geq 0$ *of distinct primes, and 0 otherwise.*

Proof. For each prime $p \in \mathbb{N}$, let X_p be the set $\{1, p, p^2, \dots\}$. Then the partial order on $\mathbb{N}$ is induced from the one on the "restricted product" of the X_p's, and each set X_p is poset-isomorphic to the set $\mathbb{N}$ under the usual ordering $\leq$.

Moreover, if $n = \prod_{i=1}^r p_i^{n_i}$ for $n_i > 0$, then we see that

$$\mu_{\mathbb{N}}(n) = \mu(1, n) = \prod_{i=1}^{r} \mu_{p_i}(1, p_i^{n_i})$$

Therefore $\mu(n)$ is nonzero if and only if each $n_i \leq 1$, and then we have $\mu(n) = \prod_{i=1}^r \mu_{p_i}(1, p_i) = (-1)^r$ from earlier results. □

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