EVOLUTION OF PLANE CURVES DRIVEN BY A NONLINEAR FUNCTION OF CURVATURE AND ANISOTROPY

Karol Mikula and Daniel Ševčovič

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Abstract. In this paper we study the evolution of plane curves satisfying the geometric equation $v = \beta(k, \nu)$, where v is the normal velocity, k and ν are the curvature and tangential angle of the plane curve Γ. We follow the direct approach and we analyze the so-called intrinsic heat equation governing the motion of plane curves obeying the geometric equation. By contrast to the usual approach, the intrinsic heat equation is modified to include an appropriate nontrivial tangential velocity functional α. We also show how the presence of a tangential velocity α can prevent numerical solutions from forming various instabilities. From analytical point of view we present some new results on short time existence of a regular family of evolving curves in the case when $\beta(k, \nu) = \gamma(\nu)k^m$, $0 < m \le 2$ and the governing system of equations includes a nontrivial tangential velocity functional.

1. Introduction

The goal of this paper is to study the evolution of closed smooth plane curves $\Gamma : S^1 \to \mathbb{R}^2$ where $S^1 = \mathbb{R}/\mathbb{Z}$. We consider the case when the normal velocity v of the evolving family of plane curves $\Gamma^t, t \in (0, T)$, is a function of the curvature k and the tangential angle ν

$$v = \beta(k, \nu) \ . \tag{1.1}$$

Geometric equations of the form (1.1) are often considered as a model for various applied problems as well as they can arise in a purely mathematical context. Examples of applications include the following topics:

a) Material science dealing with *motion of interface* (or free boundary) between solid and liquid phases. Eq.(1.1) corresponds to the so-called Gibbs-Thomson law governing crystal growth in the undercooled liquid ([G], [S1], [BM]). In [AG1], [AG2], [AG3] Angenent and Gurtin studied a motion of phase interfaces. It turns out that equations like (1.1) can be used in order to describe the evolution of phase boundaries. More precisely, they proposed to study the equation of the form

$$\mu(\nu, v)v = h(\nu)k - g,$$

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where μ is the kinetic coefficient and the quantities h, g arise from the constitutive description of the phase boundary. The dependence of the normal velocity v on the curvature k is related to surface tension effects on the interface whereas the dependence on ν (orientation of interface) introduces anisotropy into the model. In general, the kinetic coefficient μ may also depend on v giving rise to a nonlinear dependence of the function $\beta(k,\nu)$ on k and ν. If interface velocities are slow then $\beta(k,\nu)$ is linear in k (c.f. [AG1]) and Eq. (1.1) corresponds to the classical anisotropic curve evolution governed by constitutive description of interface. In the isotropic case we end up with the classical (i.e linear $v = k$) curve shortening flow problem studied extensively from both the mathematical (see e.g. [GH], [AL], [A1], [Gr]) as well as numerical point of view (see [D1],[D2],[D3], [De]).

b) In the context of image processing, the so-called *morphological image and shape multiscale analysis* is often used due to its contrast and affine invariance properties. It has been introduced by Sapiro and Tannenbaum ([ST]) and Alvarez, Guichard, Lions and Morel ([AGLM], [AM]) as a natural consequence of axiomatization of computer vision theories. In general, an image silhouette (boundary of a distinguished shape) corresponds to a level line of image intensity function. The *morphological scaling* (shape analysis) of the silhouette leads to the equation of the form (1.1) without anisotropic part. Namely, *affine invariant scale space* has special conceptual meaning and importance. It is a generalization of the linear curve shortening flow, and is given by equation (1.1) where $\beta(k) = k^{1/3}$ ([AGLM], [ST], [AST]). In the problems of image segmentation ([KWT], [Ma]) also "anisotropic like" models have been used quite recently ([CKSS]). Models involving equation (1.1) are also used in other applied sciences, e.g. in modeling of flame front propagation in combustion, computing of first arrival times of seismic waves, in computational geometry, robotics, semiconductors industry, etc. For a comprehensive overview of applications we refer to [Se2].

As we have seen, in the last decade, geometric equations of the form (1.1) attracted a lot of attention in mathematics, mathematical modeling, numerical analysis and scientific computing. The analytical methods for mathematical study of (1.1) are strongly related to numerical techniques for tracing of the curve evolutions in computer simulations. From the computational point of view, two main approaches are used for solving Eq. (1.1):

1. *Direct approach* in which we seek for a parameterization of evolving curve. To this end, one can solve the so-called *intrinsic heat equation* in order to find a position vector (see e.g. [D1], [D2], [D3], [MS], [S1], [S2]), or one can find a curvature function k by solving *porous-medium like equation* ([MK], [M]). There are also other direct methods based on solving the so-called *crystalline curvature* approximation ([Gi, [GiK], [UY]), special finite difference schemes ([K1], [K2]) and a method based on *erosion of polygons* in affine invariant case ([Mo]).

2. *Level set approach* based on introducing of an auxiliary function $\phi(x,t)$, $\phi : \mathbb{R}^d \times \mathbb{R}_0^+ \to \mathbb{R}$ ($d = 2$ or 3) to be determined by solving *the level set equation* (see e.g. [OSe], [Se1], [Se2], [Se3], [HMS]). In this approach, the zero level set of the function ϕ represents the evolving curve (surface). The other indirect method is based on *phase-field formulations* (see e.g. [C], [NPV], [EPS], [BM]). The level

set approach handles implicitly the curvature driven motion passing the problem to higher dimensional space. By using the level set approach we can deal with splitting and/or merging of evolving curves in a robust way. However, in the case of curve evolution without assuming sudden topological changes, the direct approach is more efficient and straightforward computational method.

In this paper we are concerned with the direct approach only. We consider the power like function $\beta(k,\nu) = \gamma(\nu)|k|^{m-1}k$ where $\gamma(\nu) > 0$ is a given anisotropy function and $m > 0$. From analytical perspective, the main purpose is to establish short time existence of a family of regular smooth plane curves satisfying the geometric equation (1.1). The results were obtained for the case $0 < m \leq 2$.

In our approach, a family of evolving curves is represented by their position vector x satisfying the geometric equation

$$\partial_t x = \beta(k,\nu)\vec{N} + \alpha\vec{T}\ . \tag{1.2}$$

Notice that the presence of arbitrary tangential velocity functional α does not change the shape of evolving curves. The usual choice therefore is $\alpha = 0$. From the numerical point of view, such a choice of α may lead to computational instabilities caused by merging of numerical grid points representing a discrete curve or by formation of the so-called swallow tails. Here we present an appropriate choice of a nontrivial tangential velocity α. It turns out that if α is a solution of the nonlocal equation

$$\frac{\partial \alpha}{\partial s} = k\beta(k,\nu) - \frac{1}{|\Gamma|}\int_\Gamma k\beta(k,\nu)\, ds \tag{1.3}$$

then the material points are uniformly redistributed along the evolved curve. This choice of α results in a powerful numerical scheme having the property of uniform in time redistribution of grid points and preventing the computed numerical solution from forming the above mentioned instabilities. It is worth to note that Eq. (1.2) can be transformed into the one space dimensional intrinsic heat equation (see (2.2)) and the functional α can be easily resolved from (1.3). Therefore, in each time step we have to solve several linear tridiagonal systems in order to obtain a new curve position.

The outline of the paper is as follows. In Section 2 we present a governing system of PDE's. Evolution of plane curves is parameterized by solutions of the intrinsic heat equations. We discuss the effect of a nontrivial tangential velocity on numerically computed solutions. Section 3 is focused on the analysis of the system of governing equations. The aim is to set up a closed system of parabolic equations solutions of which are the curvature, the tangent angle and the local length of a plane curve. The basic theory on short time existence of classical solutions is given in Section 4. Here we consider only the case when β'_k is nondegenerate. We follow the abstract theory due to Angenent slightly modified for the case when a nontrivial tangential velocity functional is involved in the system of governing PDE's. Section 5 is devoted to the study of the singular case when $\beta(k,\nu) = \gamma(\nu)|k|^{m-1}k$, $m \neq 1$. We extend the result due to Angenent, Sapiro and Tannenbaum ($m = 1/3$) to the general fast diffusion exponent $0 < m < 1$ as well as for the slow diffusion case where $1 < m \leq 2$. Interestingly enough, we need additional geometric assumptions made on the initial curve in the slow diffusion case $1 < m \leq 2$ and we were unable to extend the results beyond $m = 2$. In Section 6 we present a suitable choice of a tangential velocity leading to a powerful numerical scheme. We show how to construct a nontrivial tangential velocity as a nonlocal curve functional in such a way that relative local length (defined as the ratio of the local length to the total length of a curve) is constant along the evolution. Numerical scheme for full space time discretization of the governing intrinsic heat equation is presented in Section 7. We derive this scheme by using so-called flowing finite volumes. In Section 8 we show several numerical solutions of the governing system of equations and we make a comparison between results obtained by considering the trivial and nontrivial tangential velocities, respectively. One can observe the importance of the presence of a suitable nontrivial tangential velocity functional in the governing system of equations for stability of numerical computations.

2. Preliminaries

Consider an embedded regular plane curve Γ which can be parameterized by a C^2 smooth function $x : S^1 \to \mathbb{R}^2$ such that $\Gamma = \mathrm{Image}(x) = \{x(u), u \in [0,1]\}$ and $|\partial_u x| > 0$. One can define the unit tangent vector $\vec{T} = \partial_u x / |\partial_u x|$ and the unit normal vector $\vec{N}$ in such a way that $\vec{T} \wedge \vec{N} = 1$ where $\vec{a} \wedge \vec{b}$ is the determinant of the 2×2 matrix with column vectors $\vec{a}, \vec{b}$. Henceforth, will denote $\vec{a} \,.\, \vec{b}$ the Euclidean inner product of two vectors. By $|\vec{a}| = (\vec{a} \,.\, \vec{a})^{1/2}$ we denote the Euclidean norm of a vector $\vec{a}$. The derivative of a function $f = f(\xi)$ with respect to ξ we will be denoted by $\partial_\xi f$. In what follows, the unit arc-length parameterization will be denoted by s. It is related to the fixed domain parameterization $u \in [0,1]$ by $ds = |\partial_u x| du$. By k we denote the signed curvature of the curve $\Gamma = \mathrm{Image}(x)$ defined as

$$k = \partial_s x \wedge \partial_s^2 x = \frac{\partial_u x \wedge \partial_u^2 x}{|\partial_u x|^3} \tag{2.1}$$

then Frenet's formulae read as follows $\vec{T}_s = k\vec{N}, \vec{N}_s = -k\vec{T}$. The angle ν of the tangential vector is given by

$$\nu = \arg(\vec{T}) \qquad \text{i.e.} \quad (\cos\nu, \sin\nu) = \partial_s x \; .$$

To describe the time evolution $\{\Gamma^t\}, t \in [0,T)$ of an initial curve $\Gamma = \Gamma^0 = \mathrm{Image}(x^0)$ we adopt the notation $\Gamma^t = \{x(u,t), u \in [0,1]\}, \quad t \in [0,T)$, where $x \in C^2(Q_T, \mathbb{R}^2)$ and $Q_T = S^1 \times [0,T)$. We will frequently identify Q_T with $[0,1] \times [0,T)$ and the space $C^l(Q_T, \mathbb{R}^2)$ with the space of C^l differentiable functions defined on $[0,1]$ and satisfying periodic boundary conditions. The main idea how to describe the family of evolving plane curves $\Gamma^t, t > 0$, satisfying the geometric Eq. (1.1) is to parameterize Γ^t by a solution $x \in C^2(Q_T, \mathbb{R}^2)$ of the so-called intrinsic heat equation

(intrinsic heat equation)

$$\frac{\partial x}{\partial t} = \frac{1}{\theta_1}\frac{\partial}{\partial s}\left(\frac{1}{\theta_2}\frac{\partial x}{\partial s}\right), \quad x(.,0) = x^0(.) \tag{2.2}$$

where θ_1, θ_2 are geometric quantities for the curve $\Gamma^t = \text{Image}(x(.,t))$, i.e. functions whose definition is independent of particular parameterization of Γ^t and such that

$$\theta_1 \theta_2 = \frac{k}{\beta(k,\nu)} \ . \tag{2.3}$$

By using (2.3) and Frenet's formulae, Eq. (2.2) can be rewritten into the following equivalent form

$$\frac{\partial x}{\partial t} = \beta \vec{N} + \alpha \vec{T}, \quad x(.,0) = x^0(.) \tag{2.4}$$

where $\beta = \beta(k,\nu)$ is the normal velocity of the evolving curve and α is the tangential velocity given by

$$\alpha = \frac{1}{\theta_1} \frac{\partial}{\partial s} \left(\frac{1}{\theta_2} \right) \ . \tag{2.5}$$

The normal component v of the velocity $\partial_t x$ is therefore equal to $\beta(k,\nu)$. Notice that by [AST, Lemma 4.1] the family $\Gamma^t = \text{Image}(x(.,t))$ parameterized by a solution x of the geometric equation (2.4) can be converted into a solution of $\partial_t x = \beta \vec{N} + \bar{\alpha} \vec{T}$ for any continuous function $\bar{\alpha}$ by changing the space parameterization of the original curve. In particular, it means that one can take $\bar{\alpha} = 0$ without changing the shape of evolving curves. On the other hand, as it can be observed from our numerical simulations, the presence of a suitable tangential velocity term $\alpha \vec{T}$ is necessary for construction of a numerical scheme which is capable of suitable redistribution of numerical grid points along computed curve.

In [MS] the authors studied the intrinsic heat equation (2.2) with $\theta_1 = \theta_2 = (k/\beta(k))^{1/2}$. In this case Eq. (2.2) has the from $\partial_t x = \partial^2_{\bar{s}} x$ where $d\bar{s} = \theta_1 ds$. Using this particular choice of θ_1, θ_2 we were able to simulate the evolution of plane convex and non-convex curves for the case where $v = |k|^{m-1}k$. Satisfactory results were obtained only for $0 < m \le 1$ whereas various numerical instabilities appeared for the case $m > 1$. The mathematical explanation for such a behavior is very simple. If $\theta_1 = \theta_2 = |k|^{\frac{m-1}{2}}$ then, by (2.5), $\alpha = \frac{m-1}{2} |k|^{m-3} k \partial_s k = \frac{1}{2} \partial_s(|k|^{m-1})$. In the case $m > 1$ numerical grid points were driven by the tangential velocity $\alpha \vec{T}$ towards pieces of the curve with the increasing curvature. It may lead to serious computational troubles. The effect of α is just opposite when $0 < m < 1$.

Another possible choice of a nontrivial tangential velocity was studied by Deckelnick in [D] for the case $\beta(k) = k$. He proposed the governing PDE in the form $\partial_t x = \partial_u^2 x / |\partial_u x|^2$. In this case $\alpha = -\partial_u(|\partial_u x|^{-1})$ and thus $\theta_2 = \theta_1^{-1}, \theta_1 = |\partial_u x|$. This algorithm also has the property of a suitable redistribution of grid points along the computed curve. Notice that θ_1, θ_2 are not geometric quantities because of their dependence on a particular parameterization.

Note that the arc-length parameterization s occurring in the intrinsic Eq. (2.2) depends on time t and its initial position u at $t = 0$ via $ds = |\partial_u x| du$. We can therefore rewrite Eq. (2.2) into the following Eulerian form

$$\frac{\partial x}{\partial t} = \frac{1}{\theta_1 |\partial_u x|} \frac{\partial}{\partial u} \left(\frac{1}{\theta_2 |\partial_u x|} \frac{\partial x}{\partial u} \right), \qquad x(.,0) = x^0(.) \ , \tag{2.6}$$

$(u,t) \in Q_T$. Eq. (2.6) seems to be a parabolic PDE for $x = x(u,t)$. However, as θ_2 may depend on the curvature, the right hand side of (2.6) may eventually contain the third order derivative term $\partial_u^3 x$. In the next section we will show how to overcome this difficulty by embedding Eq.(2.6) into a complete system of nonlinear parabolic equations.

3. Equations for geometric quantities

The goal of this section is to derive a PDE governing the evolution of the curvature k of Γ^t and some other geometric quantities where the family of regular plane curves $\Gamma^t = \mathrm{Image}(x(.,t)), t \in [0,T)$, where x is a solution to the intrinsic heat equation (2.2). These equations will be used in order to derive a-priori estimates of solutions. Notice that such an equation for the curvature is well-known for the case when $\alpha = 0$ and it reads as follows: $\partial_t k = \partial_s^2 \beta + k^2 \beta$ where $\beta = \beta(k,\nu)$ (cf. [GH], [AG1]). Here we present a brief sketch of the derivation of the corresponding equations for the case of a nontrivial tangential velocity α.

Let us denote $\vec{p} = \partial_u x$. Then by using Frenet's formulae one has

$$\begin{aligned}
\partial_t \vec{p} &= |\partial_u x| \Big((\partial_s \beta + \alpha k)\vec{N} + (-\beta k + \partial_s \alpha)\vec{T} \Big) \\
\vec{p} \,.\, \partial_t \vec{p} &= |\partial_u x| \, \vec{T} \,.\, \partial_t \vec{p} = |\partial_u x|^2 (-\beta k + \partial_s \alpha) \\
\vec{p} \wedge \partial_t \vec{p} &= |\partial_u x| \, \vec{T} \wedge \partial_t \vec{p} = |\partial_u x|^2 \; (\partial_s \beta + \alpha k) \\
\partial_t \vec{p} \wedge \partial_u \vec{p} &= -|\partial_u x| \partial_u |\partial_u x| \; (\partial_s \beta + \alpha k) + |\partial_u x|^3 \; (-\beta k + \partial_s \alpha)
\end{aligned} \tag{3.1}$$

because $p_u = \partial_u^2 x = \partial_u(|\partial_u x| \; \vec{T}) = \partial_u |\partial_u x| \; \vec{T} + k |\partial_u x|^2 \; \vec{N}$. Since $\partial_u(\vec{p} \wedge \partial_t \vec{p}) = \partial_u \vec{p} \wedge \partial_t \vec{p} + \vec{p} \wedge \partial_u \partial_t \vec{p}$ we have $\vec{p} \wedge \partial_u \partial_t \vec{p} = \partial_u(\vec{p} \wedge \partial_t \vec{p}) + \partial_t \vec{p} \wedge \partial_u \vec{p}$. As $k = (\vec{p} \wedge \partial_u \vec{p}) \, |\vec{p}|^{-3}$ (see (2.1)) we obtain

$$\begin{aligned}
\partial_t k &= -3|p|^{-5} (\vec{p} \,.\, \partial_t \vec{p})(\vec{p} \wedge \partial_u \vec{p}) + |\vec{p}|^{-3} \left((\partial_t \vec{p} \wedge \partial_u \vec{p}) + (\vec{p} \wedge \partial_u \partial_t \vec{p}) \right) \\
&= -3k |\vec{p}|^{-2} (\vec{p} \,.\, \partial_t \vec{p}) + 2|\vec{p}|^{-3} (\partial_t \vec{p} \wedge \partial_u \vec{p}) + |\vec{p}|^{-3} \partial_u (\vec{p} \wedge \partial_t \vec{p})
\end{aligned}$$

Finally, by applying identities (3.1) we end up with the second order nonlinear parabolic PDE

(equation for the curvature)

$$\partial_t k = \partial_s^2 \beta + \alpha \partial_s k + k^2 \beta, \quad k(.,0) = k^0(.) \; . \tag{3.2}$$

Similarly as in (2.6) the above equation can be rewritten into the Eulerian form

$$\begin{aligned}
&\frac{\partial k}{\partial t} = \frac{1}{|\partial_u x|} \frac{\partial}{\partial u} \left(\frac{1}{|\partial_u x|} \frac{\partial}{\partial u} \beta(k,\nu) \right) + \alpha \frac{1}{|\partial_u x|} \frac{\partial k}{\partial u} + k^2 \beta(k,\nu) \\
&k(.,0) = k^0(.)
\end{aligned} \tag{3.3}$$

where $(u,t) \in Q_T$. The identities (3.1) can be used in order to derive an evolutionary equation for the local length $|\partial_u x|$. Indeed, $|\partial_u x|_t = (\partial_u x \,.\, \partial_u \partial_t x)/|\partial_u x| = (\vec{p} \,.\, \partial_t \vec{p})/|\partial_u x|$. By (3.1) we have

(local length equation)

$$\frac{\partial}{\partial t} |\partial_u x| = -|\partial_u x| \, k\beta \; + \; \frac{\partial \alpha}{\partial u}, \qquad |\partial_u x(.,0)| = |\partial_u x^0(.)| \; . \tag{3.4}$$

where $(u,t) \in Q_T$. In other words, $\partial_t ds = (-k\beta + \partial_s\alpha)ds$. By integrating Eq. (3.4) over the interval $[0,1]$ and taking into account that α satisfies periodic boundary conditions we obtain

(total length equation)

$$\frac{d}{dt}L^t + \int_{\Gamma^t} k\beta(k,\nu)ds = 0 \tag{3.5}$$

where $L^t = L(\Gamma^t)$ is the total length of the curve Γ^t, $L^t = \int_{\Gamma^t} ds = \int_0^1 |\partial_u x(u,t)|\, du$. If $k\beta(k,\nu) \geq 0$ then the evolution of plane curves parameterized by a solution of (2.2) represents a curve shortening flow, i.e. $L^{t_2} \leq L^{t_1} \leq L^0$ for any $0 \leq t_1 \leq t_2 \leq T$. The condition $k\beta(k,\nu) \geq 0$ is obviously satisfied in the case $\beta(k,\nu) = \gamma(\nu)|k|^{m-1}k$ where $m > 0$ and γ is a nonnegative anisotropy function.

The area enclosed by an embedded nonselfintersecting curve $\Gamma = \mathrm{Image}(x)$ can be computed as $A = \frac{1}{2}\int_0^1 x \wedge \partial_u x\, du$. Applying the identities (3.1) and taking into account that $0 = \int_0^1 \partial_u(x \wedge \partial_t x)\, du = \int_0^1 (\vec{p} \wedge \partial_t x + x \wedge \partial_t \vec{p})\, du$ where $\vec{p} = \partial_u x$ we obtain

(area equation)

$$\frac{d}{dt}A(t) + \int_{\Gamma^t} \beta(k,\nu)ds = 0\ . \tag{3.6}$$

If $\beta(k,\nu)$ is nonnegative along the evolution then the area is a non-increasing function of the time.

Denote by $[\partial_t, \partial_s]$ the commutator of the differential operators ∂_t and ∂_s, i.e. $[\partial_t, \partial_s] = \partial_t\partial_s - \partial_s\partial_t$. Since $ds = |\partial_u x|du$ it follows from the local length equation (3.4) that

(commutation relation)

$$[\partial_t, \partial_s] = (\beta k - \partial_s\alpha)\partial_s\ . \tag{3.7}$$

Recall that the tangential vector ν to a curve $\Gamma = \mathrm{Image}(x)$ is given by $\nu = \arg(\vec{T})$, i.e. $(\cos\nu, \sin\nu) = \partial_s x$ and hence by (3.7) we obtain $\partial_t\nu = \partial_s x \wedge \partial_t\partial_s x = \partial_s x \wedge \partial_s\partial_t x + (\beta k - \partial_s\alpha)(\partial_s x \wedge \partial_s x)$. Applying Frenet's formulae and (2.4) we obtain

(tangential vector equation)

$$\partial_t\nu = \partial_s\beta + \alpha k, \quad \nu(.,0) = \nu^0(.)\ . \tag{3.8}$$

Clearly,

$$\partial_s\nu = \partial_s x \wedge \partial_s^2 x = k\ . \tag{3.9}$$

Differentiating the curvature equation (3.2) with respect to t and taking into account (3.8) yields an equation for the normal velocity $v = \beta(k,\nu)$

(normal velocity equation)

$$\begin{aligned} &\partial_t v = \beta'_k\left(\partial_s^2 v + \alpha\partial_s k + k^2 v\right) + \beta'_\nu\left(\partial_s v + \alpha k\right) \\ &v(.,0) = v^0(.) = \beta(k^0(.), \nu^0(.)) \end{aligned} \tag{3.10}$$

where β'_k and β'_ν are partial derivatives of the function $\beta = \beta(k, \nu)$ with respect to k and ν, resp. Next we derive an equation for the gradient of the normal velocity $w = \partial_s v = \partial_s \beta(k, \nu)$. Using the commutation relation (3.7) we have

$$\begin{aligned}\partial_t w &= \partial_t \partial_s v = \partial_s \partial_t v + (vk - \partial_s \alpha)\partial_s v \\ &= \partial_s \left(\beta'_k(\partial_s w + \alpha \partial_s k + k^2 v)\right) + \beta'_\nu(w + \alpha k) + (vk - \partial_s \alpha) w \ .\end{aligned}$$

Since

$$w = \partial_s \beta(k, \nu) = \beta'_k \partial_s k + \beta'_\nu \partial_s \nu = \beta'_k \partial_s k + \beta'_\nu k \tag{3.11}$$

we end up with an equation for the gradient w of the velocity v

(equation for the gradient of velocity)

$$\begin{aligned}&\partial_t w = \partial_s \left(\beta'_k \partial_s w\right) + \alpha \partial_s w + \partial_s \left(\beta'_k k^2 v + \beta'_\nu w\right) + kvw \\ &w(.,0) = w^0(.) = \partial_s v^0(.) \ .\end{aligned} \tag{3.12}$$

Now we are in a position to derive a closed system of governing equations for the geometric motion satisfying Eq. (1.2). It follows from (3.9) and (3.11) that $\partial_t \nu = \beta'_k \partial_s^2 \nu + k(\alpha + \beta'_\nu)$. Denoting $g = |\partial_u x|$ we can rewrite Eqs. (3.3), (3.4) and (3.8) into the following closed form

(governing equations)

$$\begin{aligned}\frac{\partial k}{\partial t} &= \frac{1}{g}\frac{\partial}{\partial u}\left(\frac{1}{g}\frac{\partial}{\partial u}\beta(k,\nu)\right) + \frac{\alpha}{g}\frac{\partial k}{\partial u} + k^2 \beta(k,\nu) \\ \frac{\partial \nu}{\partial t} &= \frac{\beta'_k(k,\nu)}{g}\frac{\partial}{\partial u}\left(\frac{1}{g}\frac{\partial \nu}{\partial u}\right) + k(\alpha + \beta'_\nu(k,\nu)) \\ \frac{\partial g}{\partial t} &= -gk\beta(k,\nu) + \frac{\partial \alpha}{\partial u}\end{aligned} \tag{3.13}$$

$(u,t) \in [0,1] \times (0,T)$. A solution to (3.13) is subject to the initial conditions

$$k(.,0) = k^0, \quad \nu(.,0) = \nu^0, \quad g(.,0) = g^0 \ . \tag{3.14}$$

and periodic boundary conditions. Notice that the initial conditions for k^0, ν^0, g^0 are related through the identity

$$\partial_u \nu^0 = g^0 k^0 \ . \tag{3.15}$$

In general, the function $\alpha = \alpha(k, \nu, g)$ is a nonlinear function which will be determined later. In Section 6 of this paper we present the choice of α leading to a powerful numerical algorithm preserving relative local length between numerical grid points.

4. Short time existence of solutions in the nondegenerate case

In this section we prove short time existence of a classical solution of the governing system of Eqs. (3.13) by using the abstract result due to Angenent (cf. [A4]).

Denote $\Phi = (k, \nu, g)^T$. Then Eq. (3.13) can be rewritten as a fully nonlinear PDE of the form

$$\partial_t \Phi = f(\Phi), \qquad \Phi(0) = \Phi^0 \tag{4.1}$$

where $f(\Phi) = F(\Phi, \alpha(\Phi))$ and $F(\Phi, \alpha)$ is the right hand side of (3.13). Suppose that $\beta = \beta(k, \nu)$ is a C^2 smooth function and such that

$$0 < \lambda_- \le \beta'_k(k, \nu) \le \lambda_+ < \infty, \quad \text{for any } k, \nu \tag{4.2}$$

where $\lambda_\pm > 0$ are constants and β'_k is a partial derivative of β w.r. to k. Given $0 < \sigma < 1$ we denote by $E_0, E_{1/2}, E_1$ the following Banach spaces

$$\begin{aligned} E_0 &= c^\sigma(S^1) \times c^\sigma(S^1) \times c^{1+\sigma}(S^1) \\ E_{1/2} &= c^{1+\sigma}(S^1) \times c^{1+\sigma}(S^1) \times c^{1+\sigma}(S^1) \\ E_1 &= c^{2+\sigma}(S^1) \times c^{2+\sigma}(S^1) \times c^{1+\sigma}(S^1) \end{aligned} \tag{4.3}$$

where $c^{k+\sigma}, k = 0, 1, 2$, is the little Hölder space, i.e., the closure of $C^\infty(S^1)$ in the topology of the Hölder space $C^{k+\sigma}(S^1)$ (see [A2]). Let $\mathcal{O}_i \subset E_i$ be an open subset in E_i such that $g > 0$ for any $(k, \nu, g)^T \in \mathcal{O}_i$, $i = \frac{1}{2}, 1$. If we assume

$$\alpha \in C^1(\mathcal{O}_{1/2}, c^{2+\sigma}(S^1)) \tag{4.4}$$

then the mapping f is a smooth mapping from $\mathcal{O}_1 \subset E_1$ into E_0.

If the Fréchet derivative $df(\bar{\Phi}) \in \mathcal{L}(E_1, E_0)$ belongs to the maximal regularity class $\mathcal{M}_1(E_0, E_1)$ for any $\bar{\Phi} \in \mathcal{O}_1$ then, by [A4, Theorem 2.7], Eq. (4.1) has a unique solution $\Phi \in Y^T = C([0, T], E_1) \cap C^1([0, T], E_0)$ on some small enough interval $[0, T]$. Recall that the class $\mathcal{M}_1(E_0, E_1) \subset \mathcal{L}(E_1, E_0)$ consists of those generators of analytic semigroups $A : D(A) = E_1 \subset E_0 \to E_0$ for which the linear equation $\partial_t \Phi = A\Phi + h(t)$, $0 < t \le 1$, $\Phi(0) = \Phi^0$, has a unique solution $\Phi \in Y^1$ for any $h \in C([0, 1], E_0)$ and $\Phi^0 \in E_1$. In other words, (E_0, E_1) is a maximal parabolic regularity pair.

Theorem 4.1. *Assume that $(k^0, \nu^0, g^0)^T \in \mathcal{O}_1 \subset E_1$ where k^0 is the curvature, ν^0 is the tangential vector and $g^0 = |\partial_u x^0| > 0$ is the local length element of the initial regular curve $\Gamma^0 = \mathrm{Image}(x^0)$. If $\beta = \beta(k, \nu)$ is a C^2 smooth function satisfying (4.2) and α obeys (4.4) then there exists a unique classical solution $\Phi = (k, \nu, g)^T \in C([0, T], E_1) \cap C^1([0, T], E_0)$ of the governing system of Eqs. (3.13) defined on some small time interval $[0, T]$. Moreover, if Φ is a maximal solution defined on $[0, T_{max})$ and $T_{max} < \infty$ then $\max |k(., t)| \to \infty$ as $t \to T_{max}$.*

P r o o f. Let $\Phi^0 \in \mathcal{O}_1$ where $\mathcal{O}_1 \subset E_1$ is an open and bounded subset of E_1, $g > 0$ for any $(k, \nu, g)^T \in \mathcal{O}_1$. The linearization of f at $\bar{\Phi} = (\bar{k}, \bar{\nu}, \bar{g})^T \in \mathcal{O}_1$ has the form $df(\bar{\Phi}) = d_\Phi F(\bar{\Phi}, \bar{\alpha}) + d_\alpha F(\bar{\Phi}, \bar{\alpha})\, d_\Phi \alpha(\bar{\Phi})$ where $\bar{\alpha} = \alpha(\bar{\Phi})$ and

$$\begin{aligned} d_\Phi F(\bar{\Phi}, \bar{\alpha})\Phi &= \partial_u(\bar{D}\partial_u \Phi) + \bar{B}\partial_u \Phi + \bar{C}\Phi, \\ d_\alpha F(\bar{\Phi}, \bar{\alpha})\alpha &= \left(\alpha \bar{g}^{-1}\partial_u \bar{k},\ \alpha \bar{k},\ \partial_u \alpha\right)^T \end{aligned}$$

$\bar{D} = \mathrm{diag}(\bar{D}_{11}, \bar{D}_{22}, 0)$, $\bar{D}_{11} = \bar{D}_{22} = \beta'_k(\bar{k}, \bar{\nu})\bar{g}^{-2} \in C^{1+\sigma}(S^1)$ and $\bar{B}, \bar{C}$ are 3×3 matrices with $C^\sigma(S^1)$ smooth coefficients, $\bar{B}_{3j} = 0, \bar{C}_{3j} \in C^{1+\sigma}$. By (3.9) we have $g^{-1}\partial_u \beta(k,\nu) = g^{-1}\beta'_k \partial_u k + \beta'_\nu k$ and so the principal part is indeed a diagonal one. The linear operator $A_1 = \partial_u(\bar{D}\partial_u \Phi)$, $D(A_1) = E_1$ is a generator of an analytic semigroup on E_0 and moreover $A_1 \in \mathcal{M}_1(E_0, E_1)$ (cf. [A4]). Notice that $d_\alpha F(\bar{\Phi}, \bar{\alpha})$ belongs to $\mathcal{L}(C^{2+\sigma}(S^1), E_{1/2})$ and this is why we can write $d_\Phi f(\bar{\Phi})$ as the sum $A_1 + A_2$ where $A_2 \in L(E_{1/2}, E_0)$, $\|A_2\Phi\|_{E_0} \le C\|\Phi\|_{E_{1/2}} \le C\|\Phi\|_{E_0}^{1/2}\|\Phi\|_{E_1}^{1/2}$ is a relatively bounded linear perturbation of A_1 with zero relative bound (cf. [A4]). Since the class $\mathcal{M}_1$ is closed w.r. to such perturbations (see [A4, Lemma 2.5]) we have $d_\Phi f(\bar{\Phi}) \in \mathcal{M}_1(E_0, E_1)$. The proof of the short time existence of a solution Φ now follows from [A4, Theorem 2.7].

Finally, we will show that the maximal curvature becomes unbounded as $t \to T_{max} < \infty$. Suppose to the contrary that $\max_{\Gamma^t} |k(.,t)| \le M < \infty$ for any $t \in [0, T_{max})$. According to [A2, Theorem 3.1] there exists a unique maximal solution $\Gamma : [0, T'_{max}) \to \Omega(\mathbb{R}^2)$ satisfying $\Gamma(0) = \Gamma^0$ and the geometric equation (1.1). Recall that $\Omega(\mathbb{R}^2)$ is the space of C^1 regular curves in the plane (cf. [A2]). Moreover, $\Gamma(t)$ is a C^∞ smooth curve for any $t \in (0, T'_{max})$ and the maximum of the absolute value of the curvature tends to infinity as $t \to T'_{max}$. Thus $T_{max} < T'_{max}$ and therefore the curvature and subsequently ν remain bounded in $C^{2+\sigma'}$ norm on the interval $[0, T_{max}]$ for any $\sigma' \in (\sigma, 1)$. Applying compactness arguments one sees that the limit $\lim_{t \to T_{max}} \Phi(.,t)$ exists and remains bounded in the space E_1 and one can continue the solution Φ beyond T_{max}, a contradiction. $\square$

Next we will show how to construct a classical solution $x = x(u,t)$ of the intrinsic heat equation (2.2). Suppose that $\tilde{\Phi} = (\tilde{k}, \tilde{\nu}, \tilde{g})^T$ is a classical solution of the system (3.13) existing on the time interval $[0,T]$. Let us construct a flow of plane curves $\Gamma^t = \mathrm{Image}(x(.,t)), t \in [0,T]$, as follows

$$x(u,t) = x^0(u) + \int_0^t (\tilde{\beta}\tilde{\vec{N}} + \tilde{\alpha}\tilde{\vec{T}})\, d\tau \tag{4.5}$$

where $\tilde{\vec{N}} = (-\sin\tilde{\nu}, \cos\tilde{\nu})^T$, $\tilde{\vec{T}} = (\cos\tilde{\nu}, \sin\tilde{\nu})^T$, $\tilde{\beta} = \beta(\tilde{k}, \tilde{\nu})$ and $\tilde{\alpha} = \alpha(\tilde{k}, \tilde{\nu}, \tilde{g})$. We claim that $x(u,t)$ is a classical solution of (2.2).

Theorem 4.2. *Assume β and α satisfy assumptions of Theorem 4.1. Let $\tilde{\Phi} = (\tilde{k}, \tilde{\nu}, \tilde{g})^T$ be a classical solution of (3.13) such that the quantities $\tilde{k}, \tilde{\beta}$ and $\tilde{g}^{-1}\partial_u\tilde{\alpha}$ are bounded. Then $x = x(u,t)$ given by (4.5) satisfies $|\partial_u x| = \tilde{g}, k = \tilde{k}, \nu = \tilde{\nu}, \vec{N} = \tilde{\vec{N}}, \vec{T} = \tilde{\vec{T}}$ where $k, \nu, \vec{N}, \vec{T}$ represent the curvature, the tangent angle, the unit normal and tangent vectors of the curve $\Gamma^t = \mathrm{Image}(x(.,t))$. Moreover, $x \in C([0,T]; (C^{2+\sigma}(S^1))^2) \cap C^1([0,T]; (C^\sigma(S^1))^2)$ is a classical solution of the intrinsic heat equation (2.2).*

P r o o f. First we prove that $\partial_u\tilde{\nu} = \tilde{g}\tilde{k}$ for any classical solution of (3.13). Indeed, if we denote $K = \tilde{k} - \tilde{g}^{-1}\partial_u\tilde{\nu}$ then it is an easy calculus to verify that K satisfies the linear parabolic equation

$$\frac{\partial K}{\partial t} = \frac{1}{\tilde{g}}\frac{\partial}{\partial u}\left(\frac{\beta'_k(\tilde{k}, \tilde{\nu})}{\tilde{g}}\frac{\partial K}{\partial u}\right) - \frac{1}{\tilde{g}}\frac{\partial}{\partial u}\left(\beta'_\nu(\tilde{k}, \tilde{\nu})K\right) + \left(\tilde{k}\tilde{\beta} - \frac{1}{\tilde{g}}\frac{\partial\tilde{\alpha}}{\partial u}\right)K\ .$$

Moreover, $K(u,0)=0$ because $\partial_u\tilde\nu^0=\tilde g^0\tilde k^0$ (see (3.15)). The term $\tilde k\tilde\beta-\tilde g^{-1}\partial_u\tilde\alpha$ is assumed to be bounded and therefore we may conclude that $K(u,t)=0$ for any $u\in[0,1], t\in[0,T]$. As $\tilde g\tilde k=\partial_u\tilde\nu$ we end up with Frenet's formulae $\partial_u\tilde{\vec T}=\tilde g\tilde k\tilde{\vec N}$ and $\partial_u\tilde{\vec N}=-\tilde g\tilde k\tilde{\vec T}$. Similarly as in the proof of the identities (3.1) the equation $\partial_t x=\tilde\beta\tilde{\vec N}+\tilde\alpha\tilde{\vec T}$ yields $p.\partial_t p=\tilde g^2(-k\tilde\beta-\tilde g^{-1}\partial_u\tilde\alpha)$ where $p=\partial_u x$. Thus $\partial_t(|p|^2)=2p.\partial_t p=2\partial_t(\tilde g^2)$ and therefore $|\partial_u x|=|p|=\tilde g$ because $|\partial_u x^0|=g^0$. Again, using the last two equations in (3.1) we obtain $k=(\partial_u x\wedge\partial_u^2 x)/|\partial_u x|^3=\tilde k$ and subsequently $\nu=\tilde\nu$ which gives us $\vec N=\tilde{\vec N},\vec T=\tilde{\vec T}$. Hence $x=x(u,t)$ obeys Eq.(2.4), i.e. $\partial_t x=\beta(k,\nu)\vec N+\alpha(k,\nu,g)\vec T$. Therefore x is a solution of the intrinsic heat equation (2.2). The regularity properties of x follows directly from the regularity of the solution $\tilde\Phi$ and (4.5) (see Theorem 4.1). $\square$

5. Analysis of the equations for geometric quantities and short time existence of solutions in the degenerate case.

The aim of this section is to prove the short time existence of smooth solutions of the curve shortening flow governed by the intrinsic heat equation (2.2). Throughout the rest of the paper we will assume that the normal velocity function $v=\beta(k,\nu)$ has the form

$$\beta(k,\nu)=\gamma(\nu)|k|^{m-1}k$$

where $m>0$ and $\gamma:\mathbb{R}\to\mathbb{R}^+$ is a given C^∞ smooth anisotropy function satisfying

$$0<C_1^{-1}\le\gamma(\nu)\le C_1,\quad |\gamma'_\nu(\nu)|\le C_1\qquad \text{for any } \nu\in\mathbb{R}\tag{5.0}$$

where $C_1>0$ is a constant.

The assumptions guaranteeing the local existence of classical solutions of a nonlinear curve shortening flow developed by Angenent in [A2, A3] as well as those of Theorem 4.1 do not directly apply to the case $\beta(k,\nu)=\gamma(\nu)|k|^{m-1}k$ for $m\neq1$. Recall that these theories require β to satisfy (4.2). To make use of the result established in Theorem 4.1 we must go through a regularization argument. A similar technique was applied in the paper by Angenent, Sapiro and Tannenbaum [AST] for the case of an isotropic function $\beta(k)=k^{1/3}$. In what follows, we will slightly modify their approach for more general anisotropic power like function $\beta(k,\nu)$ and for the case when the curvature equation involves a nontrivial tangential velocity term α.

Henceforth, we denote by C_i, M_i, any generic positive constant independent on the regularization parameter $0<\varepsilon\le1$. Constants M_i may depend on the initial curve Γ^0. We make the following regularization assumption on the function β.

(regularization of β)
There is a family of nondecreasing C^∞ functions $\beta^\varepsilon, 0<\varepsilon\le1$, such that

i) $\beta^\varepsilon(k,\nu)\to\beta^0(k,\nu)=\beta(k,\nu)$ as $\varepsilon\to0^+$ locally uniformly w.r.t. $(k,\nu)\in\mathbb{R}^2$;

ii) $|\beta^\varepsilon(k,\nu)|\le C_2(1+|k|^m)$ for any $k,\nu\in\mathbb{R}$.

iii) There exist constants $\lambda^\varepsilon_\pm=\lambda^\varepsilon_\pm(M_1)>0$ such that $\lambda^\varepsilon_-\le{\beta^\varepsilon_k}'(k,\nu)\le\lambda^\varepsilon_+$ and

iv) $|\beta_k^{\varepsilon}{}'(k,\nu)k^4(\beta^\varepsilon(k,\nu))^2| + \frac{|\beta_\nu^{\varepsilon}{}'(k,\nu)|^2}{\beta_k^{\varepsilon}{}'(k,\nu)} \le C_3(M_1)$ for any $|k| \le M_1$ and $\nu \in \mathbb{R}$,

It is easy to verify that the regularization family (β^ε) defined as

$$\beta^\varepsilon(k,\nu) = m\gamma(\nu)\int_0^k (\varepsilon^2+\xi^2)^{\frac{m-1}{2}}\,d\xi \qquad \text{if } \ 0<m\le 1;$$
$$\beta^\varepsilon(k,\nu) = \beta(k,\nu) + \varepsilon k \qquad \text{if } \ m>1$$

satisfies the above assumptions i)-iv) with constants $\lambda_\pm^\varepsilon > 0$ given by

$$\begin{aligned} &\lambda_-^\varepsilon = \delta, \quad \lambda_+^\varepsilon = C_1 m\varepsilon^{m-1} \qquad \text{if } \ 0<m\le 1;\\ &\lambda_-^\varepsilon = \varepsilon, \quad \lambda_+^\varepsilon = 1 + \max_{|k|\le M_1} \beta_k'(k,\nu) \qquad \text{if } \ m>1 \end{aligned} \tag{5.1}$$

where $\delta > 0$ is a constant independent of $0<\varepsilon\le 1$. Furthermore,

$$0 \le \frac{\beta^\varepsilon(k,\nu)}{k} \le \max(1, m^{-1})\,\beta_k^{\varepsilon}{}' \quad \text{for any } k,\nu\in\mathbb{R} \ \text{ and } 0<\varepsilon\le 1\ . \tag{5.2}$$

Let us emphasize the fact that the tangential velocity α may also depend on the regularization parameter ε, i.e. $\alpha = \alpha^\varepsilon$. For instance, α^ε may depend on k and $\beta^\varepsilon = \beta^\varepsilon(k,\nu)$. Concerning the structural properties of α^ε we make the following hypotheses:

$$\sup_{\Phi\in B_{1/2}} \{|\alpha^\varepsilon| + |\partial_s\alpha^\varepsilon|; \quad \alpha^\varepsilon = \alpha^\varepsilon(\Phi),\ 0\le\varepsilon\le 1\} < \infty \tag{5.3}$$

for any set $B_{1/2} = \{(k,\nu,g)^T \in \mathcal{O}_{1/2},\ |k|\le M_1\}$ where $M_1>0$, and

$$\|\alpha^\varepsilon(k,\nu,g)\|_{C^2(S^1)} \le C\left(1 + \|g\|_{C^1(S^1)} + \|\beta^\varepsilon(k,\nu)\|^q_{C^1(S^1)}\right) \tag{5.4}$$

for any $\Phi = (k,\nu,g)^T \in B_{1/2}$ where $C = C(M_1) > 0$ is a constant and $1\le q<\frac{4}{3}$. In Section 6 we will show how to construct a so-called tangential velocity preserving the relative local length satisfying the above hypotheses.

Let Γ^0 be a smooth initial curve such that $\Phi^0 = (k^0,\nu^0,g^0)^T \in \mathcal{O}_1 \subset E_1$. By $\Phi_\varepsilon = (k_\varepsilon,\nu_\varepsilon,g_\varepsilon)^T$ we denote the classical solution of the governing system of Eqs. (3.13) with $\beta = \beta^\varepsilon$ and $\alpha = \alpha^\varepsilon$. The short time existence of Φ_ε has been justified by Theorem 4.1 for any $0<\varepsilon\le 1$. From (4.5) and Theorem 4.2 we furthermore know that the function

$$x_\varepsilon(u,t) = x^0(u) + \int_0^t \left(\beta^\varepsilon \vec{N}_\varepsilon + \alpha^\varepsilon \vec{T}_\varepsilon\right) d\tau$$

is a classical solution of the intrinsic heat equation (2.2) for any $0<\varepsilon\le 1$.

First we will show that the maximum of $|k|$ remains bounded in a short time interval $[0,T]$ and the parameterization of the curve Γ^t is regular.

Lemma 5.1. *Suppose that the regularization assumptions i), ii) are satisfied. Then there exist constants $T > 0$ and $M_1 > 0$ such that*

$$\max_{\Gamma^t} |k_\varepsilon(.,t)| \le M_1 \quad \textit{for any } t \in [0,T] \textit{ and } \varepsilon \in (0,1] \ .$$

If α^ε satisfies (5.3) then there are constants $g_\pm > 0$ such that

$$0 < g_- < g_\varepsilon(u,t) < g_+ < \infty \quad \textit{for any } (u,t) \in Q_T \textit{ and } \varepsilon \in (0,1] \ .$$

P r o o f. The proof of the first part is essentially the same as that of [AST, Theorem 6.2]. Indeed, as $\partial_t k_\varepsilon = \partial_s^2 \beta^\varepsilon + \alpha^\varepsilon \partial_s k_\varepsilon + k_\varepsilon^2 \beta^\varepsilon$ then by applying a maximum principle argument we get $\partial_t(\max_{\Gamma^t_\varepsilon} |k_\varepsilon(.,t)|) \le F^\varepsilon(\max_{\Gamma^t_\varepsilon} |k_\varepsilon(.,t)|)$ where $F^\varepsilon(k) = \max_\nu k^2 |\beta^\varepsilon(k,\nu)| \le C_2 k^2 (1 + |k|^m)$ for any $0 < \varepsilon \le 1$. Solving this differential inequality we conclude the proof of the bound for the total variation of the curvature. To prove estimates on g we integrate the third equation in (3.14) w.r. to time. We obtain $g_\varepsilon(u,t) = g^0(u)\exp(\int_0^t (-k_\varepsilon \beta^\varepsilon + g_\varepsilon^{-1} \partial_u \alpha^\varepsilon)\, d\tau)$ where $\beta^\varepsilon = \beta^\varepsilon(k_\varepsilon, \nu_\varepsilon)$. The proof now follows from the fact that both $k_\varepsilon \beta^\varepsilon$ and $g_\varepsilon^{-1} \partial_u \alpha^\varepsilon = \partial_s \alpha^\varepsilon$ are bounded for $|k| \le M_1$ and $0 < g^0 < \infty$ uniformly with respect to $\varepsilon \in (0,1]$. $\square$

Lemma 5.2. *Assume $1 < m \le 2$. Suppose that the initial curve Γ^0 satisfies*

$$\int_{\Gamma^0} \frac{k^0}{\beta(k^0, \nu^0)}\, ds < \infty \ . \tag{5.5}$$

Then there exists a constant $M_2 > 0$ such that

$$\max_{t\in[0,T]} \int_{\Gamma^t} \frac{k_\varepsilon}{\beta^\varepsilon(k_\varepsilon, \nu_\varepsilon)}\, ds + \int_0^T \int_{\Gamma^t} |\partial_s k_\varepsilon|^2\, ds \le M_2 \quad \textit{for any} \ \ 0 < \varepsilon \le 1 \ . \tag{5.6}$$

P r o o f. Denote $v = \beta^\varepsilon(k_\varepsilon, \nu_\varepsilon)$, $k = k_\varepsilon$ and $\nu = \nu_\varepsilon$. By using (3.4), the curvature equation (3.2) and the velocity equation (3.10) we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\Gamma^t} \frac{k}{v}\, ds &= \int_{\Gamma^t} \frac{\partial}{\partial t}\left(\frac{k}{v}\right) + \frac{k}{v}(-kv + \partial_s \alpha)\, ds \\ &= \int_{\Gamma^t} \left(\frac{1}{v} - \frac{k{\beta_k^\varepsilon}'}{v^2}\right)\left(\partial_s^2 v + \alpha \partial_s k + k^2 v\right) \\ &\qquad - \frac{k{\beta_\nu^\varepsilon}'}{v^2}(\partial_s v + \alpha k) - k^2 + \frac{k}{v}\partial_s \alpha\, ds \\ &= \int_{\Gamma^t} \left(\frac{1}{v} - \frac{k{\beta_k^\varepsilon}'}{v^2}\right)\partial_s^2 v - \frac{k^3 {\beta_k^\varepsilon}'}{v} - \frac{k{\beta_\nu^\varepsilon}'}{v^2}\partial_s v\, ds \end{aligned} \tag{5.7}$$

because of the identity $0 = \int_{\Gamma^t} \partial_s\left(\frac{\alpha k}{v}\right) ds = \int_{\Gamma^t} \frac{k}{v}\partial_s \alpha + \frac{\alpha}{v}\partial_s k - \frac{\alpha k}{v^2}\partial_s v\, ds$, (3.3) and (3.11). Recall that ${\beta_k^\varepsilon}' = \beta_k' + \varepsilon = \frac{m\beta}{k} + \varepsilon$ and therefore

$$\frac{k{\beta_k^\varepsilon}'}{v} = m + \varepsilon(1-m)\frac{k}{v}$$

Plugging the above expression into (5.7) and integration by parts yield the identity

$$\frac{d}{dt}\int_{\Gamma^t}\frac{k}{v}\,ds+(m-1)\int_{\Gamma^t}\frac{1}{v^2}\left(|\partial_s v|^2+\varepsilon\partial_s v\partial_s k-2\varepsilon\frac{k}{v}|\partial_s v|^2\right)ds$$
$$=-m\int_{\Gamma^t}k^2\,ds+\varepsilon(m-1)\int_{\Gamma^t}\frac{k^3}{v}\,ds-\int_{\Gamma^t}\frac{k{\beta_\nu^\varepsilon}'}{v^2}\partial_s v\,ds\ .$$

It follows from (3.11) that $\partial_s k=(\partial_s v-{\beta_\nu^\varepsilon}'k)/{\beta_k^\varepsilon}'$. Thus

$$\frac{d}{dt}\int_{\Gamma^t}\frac{k}{v}\,ds+(m-1)\int_{\Gamma^t}\left|\frac{1}{v}\partial_s v\right|^2\left(1+\frac{\varepsilon}{{\beta_k^\varepsilon}'}-2\varepsilon\frac{k}{v}\right)ds$$
$$=-m\int_{\Gamma^t}k^2\left(1-\frac{\varepsilon k}{v}\right)ds-\varepsilon\int_{\Gamma^t}\frac{k^3}{v}\,ds+\int_{\Gamma^t}\frac{k{\beta_\nu^\varepsilon}'}{v^2}\partial_s v\left(\frac{\varepsilon(m-1)}{{\beta_k^\varepsilon}'}-1\right)ds$$
$$\le\int_{\Gamma^t}\frac{\gamma'(\nu)}{\gamma(\nu)}\frac{k}{v}\partial_s v\left(\frac{\varepsilon(m-1)}{{\beta_k^\varepsilon}'}-1\right)ds\le mC_1^2\int_{\Gamma^t}\left|\frac{k}{v}\partial_s v\right|ds$$

because of the inequalities $\left|\frac{{\beta_\nu^\varepsilon}'}{v}\right|=\left|\frac{\gamma'(\nu)}{\gamma(\nu)}\right|\le C_1^2$, $0\le\frac{\varepsilon k}{v}\le 1$ and $0<\frac{\varepsilon}{{\beta_k^\varepsilon}'}\le 1$. Let us consider the auxiliary function ϕ defined as follows

$$\phi(k)=\frac{1}{k^2}\left(1+\frac{\varepsilon}{{\beta_k^\varepsilon}'}-\frac{2\varepsilon k}{v}\right)=\frac{1}{k^2}\left(1+\frac{\varepsilon}{m\gamma(\nu)|k|^{m-1}+\varepsilon}-\frac{2\varepsilon}{\gamma(\nu)|k|^{m-1}+\varepsilon}\right)\ .$$

It is an easy calculus to verify that if $1<m\le 2$ then there exists a constant $M_3>0$ independent of $0<\varepsilon\le 1$ and such that $\inf_{|k|\le M_1}\phi(k)\ge M_3$. Using the Cauchy-Schwartz inequality we get

$$\frac{d}{dt}\int_{\Gamma^t}\frac{k}{v}\,ds+M_3(m-1)\int_{\Gamma^t}\left|\frac{k}{v}\partial_s v\right|^2ds\le mC_1^2\int_{\Gamma^t}\left|\frac{k}{v}\partial_s v\right|ds$$
$$\le\frac{m^2C_1^4}{2(m-1)M_3}L^t+\frac{M_3(m-1)}{2}\int_{\Gamma^t}\left|\frac{k}{v}\partial_s v\right|^2ds$$

and so

$$\frac{d}{dt}\int_{\Gamma^t}\frac{k}{v}\,ds+\frac{M_3(m-1)}{2}\int_{\Gamma^t}\left|\frac{k}{v}\partial_s v\right|^2ds\le\frac{m^2C_1^4}{2(m-1)M_3}L^0\ .$$

Integrating the above inequality over the interval $[0,T]$, taking into account the inequality

$$\left|\frac{k}{v}\partial_s v\right|=\left|(m+(1-m)\frac{\varepsilon k}{v})\partial_s k+\frac{\gamma'(\nu)}{\gamma(\nu)}k^2\right|\ge|\partial_s k|-C_1^2M_1^2$$

and the initial time inequality $\int_{\Gamma^0}\frac{k}{v}\,ds=\int_{\Gamma^0}\frac{k^0}{\beta^\varepsilon(k^0,\nu^0)}\,ds\le\int_{\Gamma^0}\frac{k^0}{\beta(k^0,\nu^0)}\,ds<\infty$ we finally obtain the estimate (5.6). $\square$

Remark 5.1 The assumption (5.5) seems to be quite restrictive. It is worth to note that it is fulfilled in the case when the initial curve Γ^0 is strictly convex or in the case of a nonconvex smooth curve whose inflection points have at most $2+\frac{1}{m-1}$ order contact with their tangents. As an example one can consider the Bernoulli lemniscate $(x^2+y^2)^2=4xy$ having the third order contact with its tangents at the origin. In this example the assumption (5.5) is satisfied iff $0<m<2$.

Lemma 5.3. *For any $t \in (0,T)$ we have*

$$\frac{d}{dt}X_p(t) \leq -\int_{\Gamma^t} \beta_k^\varepsilon{}' |\partial_s(w^{p/2})|^2\, ds + M_4 p^2(1 + X_p(t)) \tag{5.8}$$

where $X_p(t) = \int_{\Gamma^t} |w|^p ds = \int_0^1 |w|^p |\partial_u x_\varepsilon|\, du$ *and* $w = \partial_s \beta^\varepsilon(k_\varepsilon, \nu_\varepsilon), \quad p \geq 1.$

P r o o f. Denote $k = k_\varepsilon, \nu = \nu^\varepsilon$. Applying the local length equation (3.4) and the equation for the gradient of velocity (3.12) we obtain

$$\begin{aligned}
\frac{d}{dt}X_p(t) &= \int_0^1 \left(\partial_t(|w|^p)|\partial_u x| + |w|^p(-|\partial_u x| k v + \partial_u \alpha^\varepsilon)\right) du \\
&= \int_{\Gamma^t} \left(p|w|^{p-2} w \partial_t w + |w|^p(-kv + \partial_s \alpha^\varepsilon)\right) ds \\
&= \int_{\Gamma^t} \Big(p|w|^{p-2} w \left[\partial_s\left(\beta_k^\varepsilon{}' \partial_s w + \beta_k^\varepsilon{}' k^2 v + \beta_\nu^\varepsilon{}' w\right) + \alpha^\varepsilon \partial_s w + kvw\right] \\
&\qquad\qquad + |w|^p(-kv + \partial_s \alpha^\varepsilon)\Big) ds \\
&= -p(p-1)\int_{\Gamma^t} |w|^{p-2}\left[\beta_k^\varepsilon{}' |\partial_s w|^2 + \beta_k^\varepsilon{}' k^2 v \partial_s w + \beta_\nu^\varepsilon{}' w \partial_s w\right] ds \\
&\qquad + (p-1)\int_{\Gamma^t} |w|^p k v\, ds
\end{aligned}$$

because $0 = \int_{\Gamma^t} \partial_s\left(|w|^p \alpha^\varepsilon\right) = \int_{\Gamma^t} p|w|^{p-2} w \partial_s w \alpha^\varepsilon + |w|^p \partial_s \alpha^\varepsilon\, ds$. Notice that the tangential velocity term α^ε is involved neither in the expression for X_p nor in $\frac{d}{dt}X_p$.

Applying the Cauchy-Schwartz inequality we get

$$|\beta_k^\varepsilon{}' k^2 v \partial_s w + \beta_\nu^\varepsilon{}' w \partial_s w| \leq \beta_k^\varepsilon{}' k^4 v^2 + \beta_k^\varepsilon{}' \frac{|\partial_s w|^2}{4} + \frac{|\beta_\nu^\varepsilon{}'|^2}{\beta_k^\varepsilon{}'}|w|^2 + \beta_k^\varepsilon{}' \frac{|\partial_s w|^2}{4}$$

and therefore

$$\begin{aligned}
\frac{d}{dt}X_p(t) &= -\frac{p(p-1)}{2}\int_{\Gamma^t} \beta_k^\varepsilon{}' |w|^{p-2}|\partial_s w|^2\, ds \\
&\quad + p(p-1)\int_{\Gamma^t} |w|^{p-2}\beta_k^\varepsilon{}' k^4 v^2 + |w|^p \frac{|\beta_\nu^\varepsilon{}'|^2}{\beta_k^\varepsilon{}'}\, ds + (p-1)\int_{\Gamma^t} |w|^p k v\, ds \\
&\leq -\int_{\Gamma^t} \beta_k^\varepsilon{}' |\partial_s(w^{p/2})|^2 ds \\
&\quad + C_3 p(p-1)(X_{p-2}(t) + X_p(t)) + (p-1)M_1^2 X_p(t)
\end{aligned}$$

because $|w|^{p-2}|\partial_s w|^2 = \frac{4}{p^2}|\partial_s(w^{p/2})|^2$ and $2p(p-1)/p^2 \geq 1$ for any $p \geq 2$. Since $X_{p-2}(t) = \int_{\Gamma^t} |w|^{p-2} \leq \int_{\Gamma^t}(1 + |w|^p) \leq L^t + X_p(t)$ for any $p \geq 2$ and $L^t \leq L^0$ (see (3.5)) we finally obtain the inequality (5.8) with a constant $M_4 > 0$ independent of $0 < \varepsilon \leq 1$ and $p \geq 2$. $\square$

Lemma 5.4. *Suppose that $0 < m \le 2$. If $1 < m \le 2$ we additionally suppose that the initial curve Γ^0 satisfies the condition (5.5). Then there is a constant $M_7 > 0$ such that*

if $0 < m \le 1$ then $\max_{\Gamma^t} |\partial_s \beta^\varepsilon(k_\varepsilon, \nu_\varepsilon)| \le M_7 t^{-\frac{3}{4}}$;

if $1 < m \le 2$ then $\max_{\Gamma^t} |\partial_s \beta^\varepsilon(k_\varepsilon, \nu_\varepsilon)| \le M_7 t^{-\frac{1}{2}}$

for any $0 < \varepsilon \le 1$ and $0 < t \le T$.

P r o o f. The key idea behind the proof of this estimate is a modification of the well known Nash-Moser iterative technique adopted to the flow of plane curves. It is similar, in spirit and technique, to that of [AST, Chapter 6] which has been applied in the case of the affine scaling parameterization, i.e. $\beta(k) = k^{1/3}$. By using the differential inequality (5.8) we will show that $\|w\|_p = X_p^{\frac{1}{p}}(t)$ is bounded uniformly w.r. to $p \ge 2$ and $0 < \varepsilon \le 1$ yielding the desired L^∞ estimate on $\partial_s \beta^\varepsilon(k_\varepsilon, \nu_\varepsilon)$.

Let us consider the case $0 < m \le 1$. First, we will prove an estimate for $X_2(t)$. By (5.1) we have ${\beta_k^\varepsilon}' \ge \delta > 0$ and $\beta^\varepsilon \le M_5$. Then

$$X_2 = \int_{\Gamma^t} |w|^2 = \int_{\Gamma^t} (\partial_s v)^2 = -\int_{\Gamma^t} v \partial_s^2 v = -\int_{\Gamma^t} \beta^\varepsilon \partial_s w$$

$$\le \frac{M_5}{\delta} \int_{\Gamma^t} \sqrt{{\beta_k^\varepsilon}'} |\partial_s w| \le \frac{M_5 (L^t)^{\frac{1}{2}}}{\delta} \left(\int_{\Gamma^t} {\beta_k^\varepsilon}' |\partial_s w|^2 \right)^{\frac{1}{2}}$$

According to (5.8) X_2 is a solution of the differential inequality

$$\frac{dX_2}{dt} \le -\frac{1}{M^2} X_2^2 + 4M_4(1 + X_2)$$

where $M = M_5 (L^0)^{\frac{1}{2}}/\delta$. By solving the above differential inequality we obtain $X_2(t) \le A_2^2 t^{-1}$ where A_2 is a constant depending only on m, Γ^0 and T.

Let $p \ge 2$. As $w = \partial_s v$ there must be a point at Γ^t where $w^{p/2}$ vanishes. From the interpolation inequality [AST, Prop. 6.1, (25)] we infer

$$X_p = \int_{\Gamma^t} |w|^p \le \left(\int_{\Gamma^t} |w|^{\frac{p}{2}} \right)^{\frac{4}{3}} \left(\int_{\Gamma^t} |\partial_s (w^{\frac{p}{2}})|^2 \right)^{\frac{1}{3}} \le X_{\frac{p}{2}}^{\frac{4}{3}} \left(\frac{1}{\delta} \int_{\Gamma^t} {\beta_k^\varepsilon}' |\partial_s (w^{\frac{p}{2}})|^2 \right)^{\frac{1}{3}} .$$

Let us consider the case $1 < m \le 2$. Again, we begin with an estimate for $X_2(t)$. For $1 < m$ the derivative ${\beta_k^\varepsilon}'$ is bounded uniformly w.r. to $0 < \varepsilon \le 1$. Since $|k| \le M_1$ we have $X_2(0) = \int_{\Gamma^0} |\partial_s \beta^\varepsilon|^2 \le M \int_{\Gamma^0} (1 + |\partial_s k^0|^2) < \infty$. Integrating the differential inequality (5.8) we obtain $X_2(t) \le 1 + X_2(t) \le (1 + X_2(0)) \exp(4M_4 t) \le A_2^2$ for any $0 \le t \le T$, where $A_2 > 0$ is a constant. By using the Cauchy-Schwartz inequality we obtain

$$\sup_{\Gamma^t} |w|^{\frac{p}{2}} \le \int_{\Gamma^t} |\partial_s (w^{\frac{p}{2}})| \le \left(\int_{\Gamma^t} \frac{1}{{\beta_k^\varepsilon}'} \right)^{\frac{1}{2}} \left(\int_{\Gamma^t} {\beta_k^\varepsilon}' |\partial_s (w^{\frac{p}{2}})|^2 \right)^{\frac{1}{2}} .$$

According to (5.2) and Lemma 5.2 we have $\left(\int_{\Gamma^t} \frac{1}{{\beta_k^\varepsilon}'} \right)^{\frac{1}{2}} \le M_6$ and hence

$$X_p = \int_{\Gamma^t} |w|^p \le \sup_{\Gamma^t} |w|^{\frac{p}{2}} X_{\frac{p}{2}} \le M_6 X_{\frac{p}{2}} \left(\int_{\Gamma^t} {\beta_k^\varepsilon}' |\partial_s (w^{\frac{p}{2}})|^2 \right)^{\frac{1}{2}}$$

In both cases, taking into account (5.8) we end up with a differential inequality

$$\frac{d}{dt}X_p(t) \leq -\hat{\delta}\frac{X_p^L(t)}{X_{p/2}^K(t)} + Mp^2(1+X_p(t)) \tag{5.9}$$

for any $t \in (0,T], p \geq 2$ and $0 < \varepsilon \leq 1$ where $(L,K) = (3,4)$ if $0 < m \leq 1$, $(L,K) = (2,2)$ if $1 < m \leq 2$ and $\hat{\delta}, M > 0$ are constants independent of $0 < \varepsilon \leq 1$ and $p \geq 2$. In the case $(L,K) = (3,4)$ this s exactly the same differential inequality as that of [AST, Eq. (2.4)]. Following the iterative method of supersolutions to the differential inequality (5.9) presented in [AST, Chapter 6], given a couple (L,K) such that $K = 2(L-1), L > 1$ one can prove the existence of a bounded sequence (A_k), $0 < A_k \leq M_7$ such that

$$X_{p_k}(t) \leq A_k^{p_k} t^{-\alpha_k p_k}$$

for any $p_k = 2^{k+1}, k \geq 0$ where $\alpha_0 = \frac{1}{2}$ for $0 < m \leq 1$, $\alpha_0 = 0$ for $1 < m \leq 2$ and

$$\alpha_{k+1} = \alpha_k + \frac{1}{(L-1)}2^{-k-2} = \alpha_0 + \frac{1}{L-1}\sum_{l=0}^{k} 2^{-l-2} \to \alpha_0 + \frac{1}{2(L-1)}$$

as $k \to \infty$. This yields the estimate

$$\sup_{\Gamma^t} |\partial_s \beta^\varepsilon| = \lim_{k\to\infty} X_{p_k}^{\frac{1}{p_k}}(t) \leq M_7 t^{-\left(\alpha_0 + \frac{1}{2(L-1)}\right)}$$

for any $t \in (0,T]$ and $0 < \varepsilon \leq 1$. Since $\alpha_0 = \frac{1}{2}, L = 3$ for $0 < m \leq 1$ and $\alpha_0 = 0, L = 2$ for $1 < m \leq 2$ the proof of Lemma follows. $\square$

Summarizing all the previous results we conclude the following a-priori estimates.

Lemma 5.5. *Assume* $0 < m \leq 2$. *Let* $\Phi_\varepsilon = (k_\varepsilon, \nu_\varepsilon, g_\varepsilon)^T$ *be a classical solution of (3.13) existing on the interval* $I = [0,T]$ *and satisfying the initial condition* $\Phi^0 \in \mathcal{O}_1 \subset E_1$. *If* $1 < m \leq 2$ *we furthermore assume that the initial curve* Γ^0 *satisfies the condition (5.5). If the tangential velocity* α^ε *satisfies the condition (5.3) then*

a) $k_\varepsilon,\ \beta^\varepsilon,\ t^{\frac{3}{4}}\partial_u\beta^\varepsilon \in L^\infty(Q_T)$;
b) $g_\varepsilon,\ g_\varepsilon^{-1} \in W^{1,\infty}(I, L^\infty(S^1))$;
c) $\partial_u\nu_\varepsilon,\ t^{\frac{3}{4}}\partial_t\nu_\varepsilon \in L^\infty(Q_T)$;
d) $x_\varepsilon \in (W^{1,\infty}(Q_T))^2$;
and if, in addition, α^ε *obeys the condition (5.4) then*
e) $g_\varepsilon, g_\varepsilon^{-1} \in W^{1,\infty}(Q_T)$ *and* $\partial_u x_\varepsilon \in (W^{1,\infty}(Q_T))^2$.
and their corresponding norms are bounded independently of $0 < \varepsilon \leq 1$.

P r o o f. The statement a) is an immediate consequence of Lemmata 5.1 and 5.4 and the assumption ii) made on the regularization β^ε. Since $\partial_t g_\varepsilon = -g_\varepsilon k_\varepsilon \beta^\varepsilon + \partial_u \alpha^\varepsilon$ and $\partial_t g_\varepsilon^{-1} = -g_\varepsilon^{-2}\partial_t g_\varepsilon$ the statement b) follows from a), Lemma 5.1 and the assumption (5.3). The bounds for ν_ε follow from the identities $\partial_u\nu_\varepsilon = g_\varepsilon k_\varepsilon, \partial_t\nu_\varepsilon = \partial_s\beta^\varepsilon + \alpha^\varepsilon k_\varepsilon$ (see (3.8), (3.9)). As $\partial_t x_\varepsilon = \beta^\varepsilon \vec{N}_\varepsilon + \alpha^\varepsilon T_\varepsilon$, $\partial_u x_\varepsilon = g_\varepsilon \vec{T}_\varepsilon$ and $\beta^\varepsilon, \alpha^\varepsilon, g_\varepsilon \in$

$L^\infty(S^1)$ we conclude the statement d). Let us assume α^ε also satisfies the condition (5.4). By integrating the third equation in (3.13) we obtain $g_\varepsilon(u,t) = g^0(u)\exp\left(\int_0^t(-k_\varepsilon\beta^\varepsilon + g_\varepsilon^{-1}\partial_u\alpha^\varepsilon)d\tau\right)$. Furthermore,

$$\partial_u(k_\varepsilon\beta^\varepsilon) = k_\varepsilon\partial_u\beta^\varepsilon + \beta^\varepsilon\partial_u k_\varepsilon = \left(k_\varepsilon + \frac{\beta^\varepsilon}{\beta_k^{\varepsilon\,\prime}}\right)\partial_u\beta^\varepsilon - \frac{\beta^\varepsilon\beta_\nu^{\varepsilon\,\prime}}{\beta_k^{\varepsilon\,\prime}}g_\varepsilon k_\varepsilon\ .$$

With regard to a) and the regularization assumption made on β^ε we can conclude $\|k_\varepsilon\beta^\varepsilon\|_{C^1} \le M\|\beta^\varepsilon\|_{C^1}$. Taking into account the condition (5.4) and Lemma 5.4 we obtain the estimate

$$\begin{aligned}
\|g_\varepsilon(.,t)\|_{C^1} &\le M\left(1+\int_0^t(\|k_\varepsilon\beta^\varepsilon\|_{C^1} + \|\alpha^\varepsilon\|_{C^2} + \|g_\varepsilon(.,\tau)\|_{C^1})d\tau\right)\\
&\le M\left(1+\int_0^t\|\beta^\varepsilon\|_{C^1}d\tau + \int_0^t\|g_\varepsilon(.,\tau)\|_{C^1}d\tau\right)\\
&\le M\left(1+\int_0^t\|g_\varepsilon(.,\tau)\|_{C^1}d\tau\right)
\end{aligned}$$

for $t\in[0,T]$ and $0<\varepsilon\le 1$. Hence the L^∞ bounds for $\partial_u g_\varepsilon$ and $\partial_u g_\varepsilon^{-1} = -g_\varepsilon^{-2}\partial_u g_\varepsilon$ follow from Gronwall's lemma. The L^∞ bounds for $\partial_u^2 x_\varepsilon$ and $\partial_t\partial_u x_\varepsilon$ now follow from the identities $\partial_u^2 x_\varepsilon = \partial_u(g_\varepsilon T_\varepsilon) = \partial_u g_\varepsilon\vec{T}_\varepsilon + g_\varepsilon^2 k_\varepsilon\vec{N}_\varepsilon$ and $\partial_t\partial_u x_\varepsilon = \partial_t(g_\varepsilon T_\varepsilon) = \partial_t g_\varepsilon\vec{T}_\varepsilon + g_\varepsilon\partial_t\nu_\varepsilon\vec{N}_\varepsilon$ and parts b), c). □

Now we are in a position to state the main result of this paper.

Theorem 5.6. *Suppose that $\beta(k,\nu) = \gamma(\nu)|k|^{m-1}k$ where $0 < m \le 2$ and γ satisfies (5.0). Let Γ^0 be a smooth regular plane curve such that $(k^0,\nu^0,g^0)^T \in \mathcal{O}_1 \subset E_1$. If $1 < m \le 2$ we also suppose that Γ^0 satisfies the condition (5.5). If the tangential velocity α^ε obeys the conditions (5.3) and (5.4) then there exists $T>0$ and a family of regular plane curves $\Gamma^t = \mathrm{Image}(x(.,t)), t\in[0,T],\ x : \overline{Q_T} = [0,1]\times[0,T]\to\mathbb{R}^2$ such that*

a) $x, \partial_u x \in (C(\overline{Q_T}))^2,\ \ \partial_u^2 x,\ \partial_t x,\ \partial_u\partial_t x \in (L^\infty(Q_T))^2$;

b) $\partial_t x.\vec{N} = \beta(k,\nu)$ for any $t\in[0,T]$ and a.e. $u\in[0,1]$ where k,ν and $\vec{N}$ are the curvatures, the tangent angle and the unit normal vector of the curve Γ^t.

P r o o f. It follows from Lemma 5.5 d) and the Ascoli-Arzelea theorem that there exists a subsequence of (x_ε) converging uniformly, i.e. $x_\varepsilon \to x$ in $(C(\overline{Q_T}))^2$ as $\varepsilon\to 0^+$. By part e) we also have $\partial_u x_\varepsilon \to \partial_u x$ in $(C(\overline{Q_T}))^2$ and $\partial_t x, \partial_u\partial_t x, \partial_u^2 x \in (L^\infty(Q_T))^2$. Again, by d) and e) we furthermore have $\nu_\varepsilon \rightrightarrows \nu, g_\varepsilon \rightrightarrows g$ in $C(\overline{Q_T})$ and $g>0$. Hence $\vec{T}_\varepsilon = g_\varepsilon\partial_u x_\varepsilon \rightrightarrows g\partial_u x = \vec{T}$ and $\vec{N}_\varepsilon \rightrightarrows \vec{N}$ where $\vec{T}$ and $\vec{N}$ are the unit tangent and normal vectors to the curve $\Gamma^t = \mathrm{Image}(x(.,t)), t\in[0,T]$. Moreover, $\arg(\vec{T}) = \nu$.

Let $t\in[0,T]$ be a fixed time instant. By a) we have $|\partial_u\beta^\varepsilon| \le M$ and, as a consequence, one has $\beta^\varepsilon \rightrightarrows \tilde\beta$ in $C(S^1)$. Denote by $b^\varepsilon : \mathbb{R}\to\mathbb{R}$ the inverse function to the increasing function $k\mapsto \beta^\varepsilon(k,\nu)/\gamma(\nu),\ \varepsilon\in[0,1], \beta^0 = \beta$. Notice that the term $\beta^\varepsilon(k,\nu)/\gamma(\nu)$ does not depend on ν. With regard to the regularization assumptions made on β^ε we have $b^\varepsilon \to b^0 = b$ locally uniformly in $\mathbb{R}$. Then for the curvature $k_\varepsilon = b^\varepsilon(\beta^\varepsilon(k_\varepsilon,\nu_\varepsilon)/\gamma(\nu_\varepsilon)) = b^\varepsilon(\beta^\varepsilon/\gamma(\nu_\varepsilon))$ we have the convergence $k_\varepsilon \rightrightarrows k = b(\tilde\beta)/\gamma(\nu)$ in $C(S^1)$. Thus $\beta(k,\nu) = \tilde\beta$. As $(\partial_u^2 x_\varepsilon)$ is bounded in $(L^\infty(S^1))^2$ we have $\partial_u^2 x_\varepsilon \overset{*}{\rightharpoonup} \partial_u^2 x$ weak star in $(L^\infty(S^1))^2$. On the other hand, $k_\varepsilon = g_\varepsilon^{-3}(\partial_u x_\varepsilon \wedge \partial_u^2 x_\varepsilon) \rightrightarrows k, g_\varepsilon^{-3} \rightrightarrows g^{-3}, \partial_u x_\varepsilon \rightrightarrows \partial_u x$. Thus $k(u,t)$ is the curvature of the curve Γ^t at the point $x = x(u,t)$ for every $t \in [0,T]$ and a.e. $u \in [0,1]$. Finally, as $(\partial_t x_\varepsilon)$ is bounded in $(L^\infty(S^1))^2$ we have $\partial_t x_\varepsilon \overset{*}{\rightharpoonup} \partial_t x$ weak star in $(L^\infty(S^1))^2$. Therefore $\beta^\varepsilon = \partial_t x_\varepsilon.\vec N_\varepsilon \overset{*}{\rightharpoonup} \partial_t x.\vec N$ as $\varepsilon \to 0^+$. Since $\beta^\varepsilon \rightrightarrows \tilde\beta = \beta(k,\nu)$ in $C(S^1)$ we conclude $\partial_t x.\vec N = \beta(k,\nu)$, as claimed. $\square$

6. Tangential velocity preserving the relative local length

As it was already mentioned in Section 2, the presence of a nontrivial tangential velocity term $\alpha\vec T$ in the governing Eq. (2.4) can prevent the numerically computed solution of (2.2) from forming numerical singularities like e.g. collapsing of grid points or formation of the so-called swallow tails. The goal of this section is to propose a suitable choice of the functional $\alpha = \alpha(k,\nu,g)$ in such a way that a numerical scheme based on this choice of α will be capable of uniform redistribution of grid points along the computed curve. The main idea behind the construction of α is to analyze the relative local length function defined as the ratio $\frac{|\partial_u x(u,t)|}{L^t}$ where L^t is the total length of Γ^t and $|\partial_u x(u,t)|$ represents the local length of Γ^t. The idea is to keep this ratio constant with respect to time, i.e.
(preservation of the relative local length)

$$\frac{d}{dt}\left(\frac{|\partial_u x(u,t)|}{L^t}\right) = 0 \tag{6.1}$$

for any $u \in [0,1]$ and $t \in I = (0,T)$. Taking into account (3.4) and (3.5) one sees that (6.1) is fulfilled iff

$$\frac{\partial\alpha}{\partial s} = k\beta(k,\nu) - \frac{1}{L}\int_\Gamma k\beta(k,\nu)\,ds \tag{6.2}$$

where $\Gamma = \Gamma^t$, $L = L(\Gamma)$, k is the curvature of Γ and β is the given normal velocity function.

In what follows, we will analyze Eq. (6.2). We will show that there exist geometric quantities θ_1, θ_2 such that the tangential velocity function α given by $\alpha = \frac{1}{\theta_1}\frac{\partial}{\partial s}\left(\frac{1}{\theta_2}\right)$ (see (2.5)) obeys Eq. (6.2). We will furthermore prove some a-priori estimates for α and $\theta_i, i = 1,2$, considered as nonlocal operators from the Banach space $E_{1/2}$ (see (4.3)) into $C^{2+\sigma}(S^1)$. First we need the following simple lemma.

Lemma 6.1. *Let β^ε be a regularization of β satisfying regularization assumptions i)-iv) from Section 5. Let $\Gamma = \mathrm{Image}(x)$ be a C^2 smooth regular plane curve. Then there exists a unique weak solution $\vartheta \in C^1(S^1)$, $\vartheta(0) = \vartheta(1) = 0$, of the equation*

$$-\frac{\partial}{\partial s}\left(\frac{\beta^\varepsilon(k,\nu)}{k}\frac{\partial\vartheta}{\partial s}\right) = k\beta^\varepsilon(k,\nu) - \frac{1}{L}\int_\Gamma k\beta^\varepsilon(k,\nu)\,ds\ . \tag{6.3}$$

Furthermore, there exists a constant $C_4 = C_4(M_1) > 0$ such that

$$\max_\Gamma |\vartheta| \le C_4 \int_\Gamma \frac{k}{\beta^\varepsilon(k,\nu)}\, ds \quad \text{and} \quad |\partial_s \vartheta| \le C_4 L(\Gamma) \frac{k}{\beta^\varepsilon(k,\nu)}$$

for any $|k| \le M_1$.

P r o o f. Denote $a = \frac{k}{\beta^\varepsilon}$, $g = |\partial_u x|$ and $f = \frac{1}{L}\int_\Gamma k\beta^\varepsilon(k,\nu)\, ds - k\beta^\varepsilon(k,\nu)$. Then $0 < a < \infty$, $g > 0$ and $a, g, f \in C(S^1)$. Hence

$$\partial_u \vartheta(u) = a(u) g(u) \left(A + \int_0^u f(v) g(v)\, dv \right) \tag{6.4}$$

for some constant A. With regard to the condition $\vartheta(0) = \vartheta(1) = 0$ we obtain the existence of a unique weak solution $\vartheta \in C^1(S^1)$ and

$$\vartheta(u) = A \int_0^u a(\xi) g(\xi)\, d\xi + \int_0^u a(\xi) g(\xi) \int_0^\xi f(v) g(v)\, dv\, d\xi$$

where $A = -\left(\int_0^1 a(\xi)g(\xi) \int_0^\xi f(v)g(v)\, dv\, d\xi\right)\left(\int_0^1 a(\xi) g(\xi)\, d\xi\right)^{-1}$. Since $ag \ge 0$ we have $|A| + |\int_0^u fg| \le 2\max_\xi \int_0^\xi |f| g \le 2\int_0^1 |f| g = 2\int_\Gamma |f|\, ds \le 4\int_\Gamma k\beta^\varepsilon(k,\nu)\, ds \le 4L(\Gamma) M_1 C_2 (1 + M_1^m) = C_4 L(\Gamma)$. This together with (6.4) yields the pointwise estimate for $|\partial_s \vartheta| = |g^{-1}\partial_u \vartheta|$. The bound for $\max|\vartheta|$ now easily follows from the boundary condition $\vartheta(0) = 0$. $\square$

Lemma 6.2. *Let $\Gamma = \mathrm{Image}(x)$ be a smooth regular plane curve such that $\Phi = (k,\nu,g)^T \in \mathcal{O}_{1/2} \subset E_{1/2}$, $|k| \le M_1$. Let β^ε be any regularization of β satisfying the regularization assumptions i)-iv) from Section 5. Then there exist geometric quantities $\theta_i^\varepsilon > 0$, $\theta_i^\varepsilon : \mathcal{O}_{1/2} \to C^1(S^1), i = 1, 2$, such that*

$$\theta_1^\varepsilon \theta_2^\varepsilon = \frac{k}{\beta^\varepsilon(k,\nu)} \quad \text{and} \quad \alpha^\varepsilon = \frac{1}{\theta_1^\varepsilon}\frac{\partial}{\partial s}\left(\frac{1}{\theta_2^\varepsilon}\right), \quad \theta_2^\varepsilon(0) = \theta_2^\varepsilon(1) = 1$$

where $\alpha^\varepsilon \in C^1(\mathcal{O}_{1/2}, C^{2+\sigma}(S^1))$ is the tangential velocity preserving the relative local length satisfying Eq. (6.2). Moreover,

$$\max_\Gamma |\theta_2^\varepsilon(\Phi)| + \max_\Gamma |\theta_2^\varepsilon(\Phi)^{-1}| \le \exp\left(M_6 \int_\Gamma \frac{k}{\beta^\varepsilon} ds\right)$$

$$|\partial_s \alpha^\varepsilon(\Phi)| \le M_7, \qquad \|\alpha^\varepsilon(\Phi)\|_{C^2} \le M_7(1 + \|\beta^\varepsilon(k,\nu)\|_{C^1} + \|g\|_{C^1}),$$

i.e. α^ε satisfies the hypothesis (5.3) and (5.4).

P r o o f. Let ϑ be a solution of Eq. (6.3). Define $\theta_2^\varepsilon = \exp(\vartheta)$ and $\theta_1^\varepsilon = k/(\beta^\varepsilon \theta_2^\varepsilon)$. The maximum bounds for θ_2^ε and $(\theta_2^\varepsilon)^{-1}$ follow from Lemma 6.1. With regard to Lemma 6.1 we obtain

$$\alpha^\varepsilon = \frac{1}{\theta_1^\varepsilon}\frac{\partial}{\partial s}\left(\frac{1}{\theta_2^\varepsilon}\right) = -\frac{\beta^\varepsilon}{k}\frac{\partial}{\partial s}\ln\theta_2^\varepsilon = -\frac{\beta^\varepsilon}{k}\frac{\partial \vartheta}{\partial s}$$

is a solution of (6.2). Since β^ε satisfies the regularization assumption we have $\alpha^\varepsilon \in C^1(\mathcal{O}_{1/2}, C^{2+\sigma}(S^1))$. Notice that the estimate for $C^{2+\sigma}$ norm of α^ε may depend on $0 < \varepsilon \leq 1$. It furthermore follows from Lemma 6.1 that $\|\alpha^\varepsilon(\Phi)\|_{C^0} \leq M_6 L(\Gamma)$. With regard to (6.2) we have $\partial_u \alpha^\varepsilon = (k\beta^\varepsilon - const)g$ where $const = \frac{1}{L}\int_\Gamma k\beta^\varepsilon ds$ is a constant. Hence $|\partial_s \alpha^\varepsilon| = g^{-1}|\partial_u \alpha^\varepsilon| \leq 2\max_\Gamma |k\beta^\varepsilon(k,\nu)| \leq M_7$. Furthermore, as $\|k\beta^\varepsilon(k,\nu)\|_{C^1} \leq M\|\beta^\varepsilon(k,\nu)\|_{C^1}$ and $|const| \leq \max_\Gamma |k\beta^\varepsilon(k,\nu)|$ we have $|\partial_u^2 \alpha^\varepsilon| \leq |const||\partial_u g| + |\partial_u(gk\beta^\varepsilon(k,\nu))| \leq M_7(1 + \|\beta^\varepsilon(k,\nu)\|_{C^1} + \|g\|_{C^1})$ and the bound for $\|\alpha^\varepsilon\|_{C^2}$ follows. This is why α^ε satisfies the assumptions (5.3) and (5.4). □

Theorem 6.3. *Suppose that $\beta(k,\nu) = \gamma(\nu)|k|^{m-1}k$ where $0 < m \leq 2$ and γ satisfies (5.0). Let $\Gamma^0 = \mathrm{Image}(x^0)$ be a smooth regular plane curve as in Theorem 5.6. Then there exists $T > 0$ and a family of regular plane curves $\Gamma^t = \mathrm{Image}(x(.,t)), t \in [0,T]$ such that*

a) $x, \partial_u x \in (C(\overline{Q_T}))^2,\ \ \partial_u^2 x,\ \partial_t x,\ \partial_u\partial_t x \in (L^\infty(Q_T))^2$;

b) the flow $\Gamma^t = \mathrm{Image}(x(.,t)), t \in [0,T]$ of regular plane curves satisfies the geometric equation

$$\partial_t x = \beta\vec{N} + \alpha\vec{T}$$

where $\beta = \beta(k,\nu)$ and α is the tangential velocity preserving the relative local length, i.e. α satisfies (6.2) and

$$\frac{|\partial_u x(u,t)|}{L^t} = \frac{|\partial_u x^0(u)|}{L^0}$$

for any $t \in [0,T]$ and $u \in [0,1]$.

P r o o f. Let us consider the tangential velocity function α^ε, $0 < \varepsilon \leq 1$, satisfying $\partial_s \alpha^\varepsilon = k_\varepsilon \beta^\varepsilon(k_\varepsilon, \nu_\varepsilon) - \frac{1}{L_\varepsilon}\int_{\Gamma_\varepsilon} k_\varepsilon \beta^\varepsilon ds$ whose existence has been verified in Lemma 6.2. Moreover, α^ε is a C^1 mapping from $\mathcal{O}_{1/2} \subset E_{1/2}$ into $c^{2+\sigma}(S^1)$ and α^ε satisfies the structural conditions (5.3) and (5.4). By Theorem 5.6 there exists a family of regular plane curves $\Gamma^t = \mathrm{Image}(x(.,t))$ with the properties as in part a). To prove b), we put $\alpha = \partial_t x.\vec{T}$ and recall that $g_\varepsilon = |\partial_u x_\varepsilon| \rightrightarrows g = |\partial_u x|$ as $\varepsilon \to 0^+$. Therefore $L^t_\varepsilon = \int_0^1 |\partial_u x_\varepsilon(u,t)|\, du \to L^t = \int_0^1 |\partial_u x(u,t)|\, du$ as $\varepsilon \to 0^+$. Thus $\frac{|\partial_u x_\varepsilon(u,t)|}{L^t_\varepsilon} \to \frac{|\partial_u x(u,t)|}{L^t}$ as $\varepsilon \to 0^+$. On the other hand, since α^ε is the tangential velocity preserving the relative local length we have $\frac{d}{dt}\frac{|\partial_u x_\varepsilon(u,t)|}{L^t_\varepsilon} = 0$. Hence $\frac{|\partial_u x^0(u)|}{L^0} = \frac{|\partial_u x(u,t)|}{L^t}$. Therefore α is the tangential velocity preserving the relative local length and from (6.1), (3.4) and (3.6) we may conclude that α satisfies Eq. (6.2) □

7. Numerical scheme

In this section we describe a numerical procedure which can be used for computing the curve evolution satisfying the geometric equation (1.1). To this end, we will propose a scheme solving the coupled system of intrinsic heat equation (2.2) for the position vector x and the equation (6.2) for the tangential velocity α. A smooth solution x is approximated by discrete plane points $x_i^j, i = 1, ..., n,\ j = 0, ..., m$ where index i represents space discretization and index j a discrete time stepping. The approximation of a curve in time $j\tau$ (with uniform time step $\tau = \frac{T}{m}$) is given by polygon with vertices $x_i^j, i = 1, ..., n$. In order to obtain such an approximation of an evolving curve in j-th time step we use the following fully discrete semi-implicit scheme

$$\frac{1}{2}(g_i^{j-1} + g_{i+1}^{j-1})\frac{x_i^j - x_i^{j-1}}{\tau} = \frac{x_{i+1}^j - x_i^j}{h_{i+1}^{j-1}} - \frac{x_i^j - x_{i-1}^j}{h_i^{j-1}}, \tag{7.1}$$

$i = 1, ..., n$, for every $j = 1, ..., m$. The coefficients in (7.1) (for simplicity we omit upper index $j-1$) are given by the following expressions

$$g_i = |r_i|\theta_{1,i}, \quad h_i = |r_i|\theta_{2,i}, \quad r_i = x_i - x_{i-1}, \quad \theta_{1,i} = \frac{k_i}{\beta_i\theta_{2,i}},$$

$$k_i = \frac{1}{2|r_i|}\mathrm{sgn}(r_{i-1} \wedge r_{i+1}) \arccos\left(\frac{r_{i+1}.r_{i-1}}{|r_{i+1}||r_{i-1}|}\right), \tag{7.2}$$

$$\nu_i = \arccos(r_{i_1}/|r_i|) \text{ if } r_{i_2} \geq 0, \quad \nu_i = 2\pi - \arccos(r_{i_1}/|r_i|) \text{ if } r_{i_2} < 0,$$

$$\beta_i = \beta^\varepsilon(k_i, \nu_i), \quad \theta_{2,i} = \exp(\vartheta_i),$$

and the system (7.1) is subject to the periodic boundary conditions $x_{i+n}^j = x_i^j$ $(i = 0, 1)$. In order to compute $\vartheta_i, i = 1, ..., n$, governing tangential redistribution of flowing points we solve

$$-\frac{\frac{\beta_i}{k_i} + \frac{\beta_{i+1}}{k_{i+1}}}{|r_i| + |r_{i+1}|}(\vartheta_{i+1} - \vartheta_i) + \frac{\frac{\beta_i}{k_i} + \frac{\beta_{i-1}}{k_{i-1}}}{|r_i| + |r_{i-1}|}(\vartheta_i - \vartheta_{i-1})$$

$$= |r_i|\left(k_i\beta_i - \left(\sum_{l=1}^n |r_l|k_l\beta_l\right)\left(\sum_{l=1}^n |r_l|\right)^{-1}\right) \tag{7.3}$$

for $i = 1, ..n$, accompanied by the periodic boundary conditions $\vartheta_{i+n} = \vartheta_i$ $(i = 0, 1)$.

The system (7.3) can be represented by a symmetric positive semidefinite tridiagonal matrix with kernel containing n-dimensional vector each component of which is equal 1. Since $\sum_{i=1}^n b_i = 0$, where b_i are the components of the right hand side of (7.3), we have assured the existence of a solution which is also unique up to an additive constant. We choose the unique solution by imposing the constraint condition $\vartheta_0 = \vartheta_n = 0$.

Then, the linear systems (7.1) can be represented by two symmetric positive definite tridiagonal matrices for which we have the existence and uniqueness of a solution. Finally, we can say that, in each discrete computational time step $j\tau$ the scheme (7.1)-(7.3) leads to solving of three tridiagonal systems, namely one for the redistribution of points along the curve and two for finding the new curve position.

Remark 7.1. The approximation (7.1) can be considered as a spatial discrete analogy of the backward Euler time discretization scheme

$$\frac{x^j - x^{j-1}}{\tau} = \frac{1}{\theta_1^{j-1}}\frac{\partial}{\partial s^{j-1}}\left(\frac{1}{\theta_2^{j-1}}\frac{x^j}{\partial s^{j-1}}\right), \quad j = 1, 2, ..., m \tag{7.4}$$

of Eq. (2.2), where the terms θ_1, θ_2 as well as arclength parameterization s are taken from the previous time step x^{j-1}, and $\Gamma^j = \mathrm{Image}(x^j)$ is considered as a smooth approximation of the evolution in discrete time $j\tau$. Denoting $\delta_\tau(x^j) = (x^j - x^{j-1})/\tau$ and $ds^{j-1} = |\partial_u x^{j-1}|du$, $\theta_1^{j-1}\theta_2^{j-1} = k^{j-1}/\beta^\varepsilon(k^{j-1}, \nu^{j-1})$ we easily obtain

$$\delta_\tau(x^j) = \tilde{\beta}\vec{N}^j + \tilde{\alpha}\vec{T}^j \tag{7.5}$$

where

$$\tilde{\beta} = \frac{|\partial_u x^j|^2 k^j}{|\partial_u x^{j-1}|^2 k^{j-1}}\beta^\varepsilon(k^{j-1}, \nu^{j-1}), \quad \tilde{\alpha} = \frac{1}{\theta_1^{j-1}}\frac{\partial}{\partial s^{j-1}}\left(\frac{1}{\theta_2^{j-1}}\frac{|\partial_u x^j|}{|\partial_u x^{j-1}|}\right).$$

Lemma 7.1. *Assume $x^{j-1} \in C^1(S^1; \mathbb{R}^2)$, $\theta_1^{j-1}, \theta_2^{j-1} \in C^1(S^1; \mathbb{R}^2)$ are such that $|\partial_u x^{j-1}| > 0, \theta_1^{j-1} > 0, \theta_2^{j-1} > 0$. Then there exists a unique solution $x^j \in C^2(S^1; \mathbb{R}^2)$ of Eq. (7.4). Moreover*

$$L^j + \tau \int_{\Gamma^j} \tilde{\beta} k^j ds^j \le L^{j-1} \tag{7.6}$$

where $L^j = \int_0^1 |\partial_u x^j| du$ represents the length of the curve Γ^j. The sequence $\Gamma^j =$ $\mathrm{Image}(x^j)$ represents a curve shortening discrete flow.

P r o o f. The existence and uniqueness of a solution x^j can be achieved in the same way as it was done in [MS, Lemma 4.1] in the case $\theta_1 = \theta_2 = k/\beta$. To prove the bound (7.6) we proceed in a similar way as in the continuous case (see (3.5)). Indeed, the time discrete analogy of the first equation in (3.1) is given by

$$\delta_\tau(\partial_u x^j) = |\partial_u x^j| \left((\partial_{s^j}\tilde{\beta} + \tilde{\alpha}k^j)\vec{N}^j + (-\tilde{\beta}k^j + \partial_{s^j}\tilde{\alpha})\vec{T}^j\right)$$

and therefore

$$\begin{aligned} |\partial_u x^j| &= \tau\delta_\tau(\partial_u x^j).\vec{T}^j + \partial_u x^{j-1}.\vec{T}^j = \tau\delta_\tau(\partial_u x^j).\vec{T}^j + |\partial_u x^{j-1}|\vec{T}^{j-1}.\vec{T}^j \\ &\le |\partial_u x^{j-1}| + \tau(-\tilde{\beta}k^j + \partial_{s^j}\tilde{\alpha})|\partial_u x^j| = |\partial_u x^{j-1}| - \tau\tilde{\beta}k^j|\partial_u x^j| + \tau\partial_u\tilde{\alpha} \end{aligned}$$

Integrating above inequality over the interval $[0, 1]$ yields the bound (7.6). Since $\beta^\varepsilon k \ge 0$ we have $\tilde{\beta}k^j \ge 0$ and therefore $L^j \le L^{j-1}$, i.e., Γ^j represents a curve shortening discrete flow. $\square$

Remark 7.2. The scheme (7.1)-(7.3) can be derived using *flowing control volume* technique (c.f. [P]). Let us consider points $x_i, i = 1, ..., n$ belonging to a smooth curve $\Gamma^t = \mathrm{Image}(x(., t))$ where x is a solution of (2.2) at time t. Denote the part of the curve connecting the points x_{i-1} and x_i by $[x_{i-1}, x_i]$. Let us consider control volume V_i around x_i consisting of part of the curve connecting centers c_i, c_{i+1} of $[x_{i-1}, x_i]$, $[x_i, x_{i+1}]$, respectively. This *centered control volume* is flowing and changing a length during the evolution respecting the new positions of the points

x_i along the curve. Let us integrate intrinsic diffusion Eq. (2.2) along the finite volume V_i. We obtain

$$\int_{V_i} \theta_1 \frac{\partial x}{\partial t} ds = \left[\frac{1}{\theta_2} \frac{\partial x}{\partial s} \right]_{c_i}^{c_{i+1}} . \tag{7.7}$$

Let us consider piecewise linear approximation of x, i.e. the polygon connecting points $x_i, i = 1, ..., n$. Then, for each $[x_{i-1}, x_i]$, which is now a line segment, we define constant geometrical quantities k_i, ν_i, β_i by the expressions in (7.2); ϑ_i can be computed numerically solving intrinsic Eq. (6.2) again by control volume approximation. Integrating (6.2) along $[x_{i-1}, x_i]$ (dual volume to V_i) we have

$$-\left[\frac{\beta}{k} \frac{\partial \vartheta}{\partial s} \right]_{x_{i-1}}^{x_i} = |r_i| \left(k_i \beta_i - \left(\sum_{l=1}^{n} |r_l| k_l \beta_l \right) \left(\sum_{l=1}^{n} |r_l| \right)^{-1} \right) . \tag{7.8}$$

Approximating $\frac{\partial \vartheta}{\partial s}(x_i)$ by $2 \frac{\vartheta_{i+1} - \vartheta_i}{(|r_i| + |r_{i+1}|)}$ and $\frac{\beta}{k}(x_i)$ by $\frac{1}{2}(\frac{\beta_i}{k_i} + \frac{\beta_{i+1}}{k_{i+1}})$ we end up with the system (7.3). Now, approximating $\frac{\partial x}{\partial t}$ by $\dot{x}_i$ inside V_i we obtain from (7.7) the system of ordinary differential equations

$$\frac{1}{2}(|r_i| \theta_{1,i} + |r_{i+1}| \theta_{1,i+1}) \dot{x}_i = \frac{1}{\theta_{2,i+1}} \frac{x_{i+1}^j - x_i^j}{|r_{i+1}|} - \frac{1}{\theta_{2,i}} \frac{x_i^j - x_{i-1}^j}{|r_i|} . \tag{7.9}$$

There is a range of possibilities how to solve this system. In order to end up with (7.1), we approximate time derivative by time difference of new and previous discrete curve positions, and all nonlinear terms are handled from the previous time step while linear terms are considered at new time level. The numerical simulations of Section 8 show that such approximation is sufficient in very general cases regarding accuracy and efficiency of computations. Moreover, using Lemma 7.1 we have guaranteed a kind of stability for numerical computations.

8. Discussion on numerical experiments

In this Section we describe numerical results obtained by the algorithm (7.1)-(7.3) for solving the geometric equation (1.1). We test properties of the model and the numerical scheme in evolution of convex as well as nonconvex (and nonrectifiable) initial curves in the presence of nonlinearity and anisotropy in the shape of function β. The effect of redistribution of discrete points representing evolving curve is documented in the same time. We have found general situations where the redistribution, based on relative local length conservation of flowing curve segments, is an important tool in correct handling of the curve evolution without other artificial operations like points removing or artificial cutting of the so-called swallow tails. The redistribution of the points representing flowing curve, based on (6.2) preserves, in a sense, initial discretization of curve and thus make its discrete representation smooth enough during evolution. First such examples are given in Figures 1a - 1b. In those experiments $\beta(k) = k$, i.e. we have classical curve shortening, and we start with initial curve with large variations in the curvature, namely

$$x_1(u) = \cos(2\pi u),$$
$$x_2(u) = \frac{1}{2} \sin(2\pi u) + \sin(x_1(u)) + \sin(2\pi u)(0.2 + \sin(2\pi u) \sin(6\pi u) \sin(6\pi u))$$

$u \in [0, 1]$, and initial discretization is given by uniform division of the range of parameter u. The curve is represented by 100 discrete points. Addition of a non-trivial tangential velocity obeying (6.2) leads to the evolution plotted in Figure 1a. In Figure 1b the points move only in the normal direction and one can see their fast merging in several regions and very poor discrete representation in other parts of the curve. In these, and also in further experiments we have used uniform time step $\tau = 0.001$. The blow up time for the curvature was $T_{max} = 0.363$. Isoperimetric ratio starting with 3.02 tends to 1.0 which is consistent with Grayson's theorem ([Gr]). In both Figures, we plot each 20-th discrete time step using discrete points representing the evolving curve and in each 60-th time step we plot also by piecewise linear curve connecting those points.

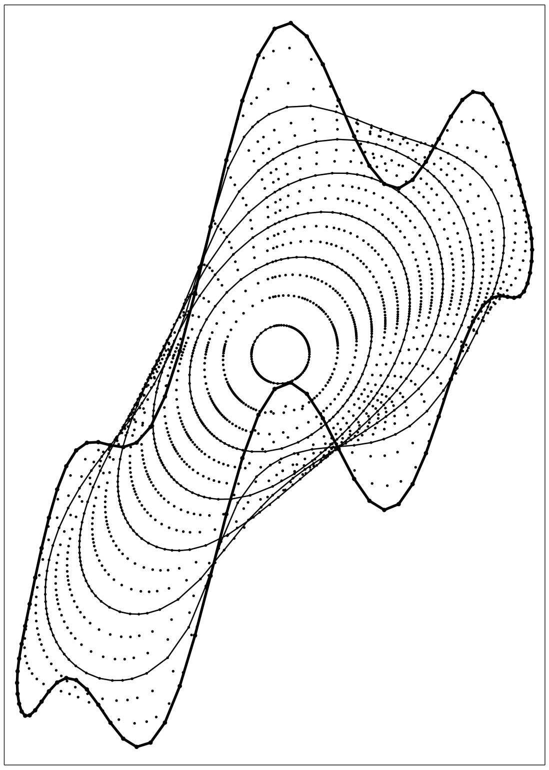

Fig. 1a. $\beta(k) = k$, discrete evolution using tangential redistribution of points.

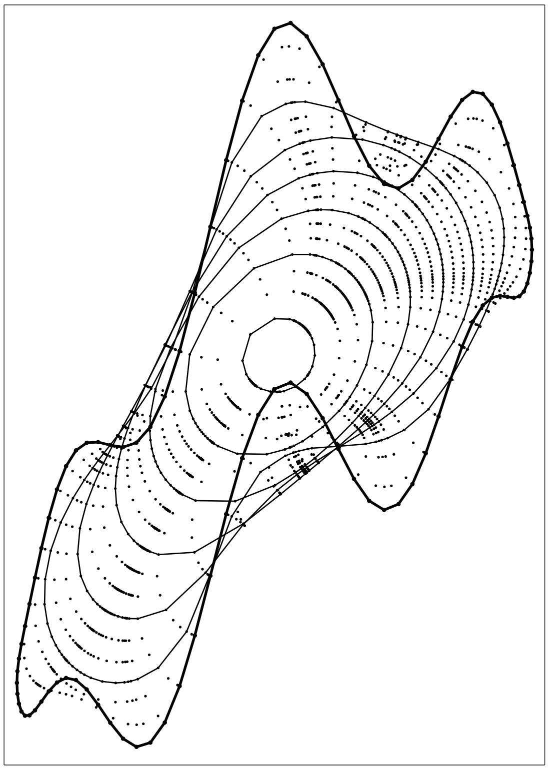

Fig. 1b. $\beta(k) = k$, without redistribution, only normal component of velocity is used.

In the next Figures 2a-2b, $\beta(k) = k^{1/3}$ i.e. we compute affine evolution of the same initial shape. The initial curve has been discretized in order to obtain a almost uniform initial distribution of discrete points. This distribution is then preserved in evolution using (6.2). We present the results in Figure 2a. The blow up time $T_{max} = 0.694$, solution tends to be ellipse (isoperimetric ratio stabilized on 1.33) which corresponds to analytical results of Sapiro and Tannenbaum ([ST]). Without points redistribution, we can see rapid merging of several points, and thus a degeneracy in $|r_i|$ (corresponding to discrete $|\partial_u x|$ term) and subsequent collapse of computation. In Figure 2b we see the evolution until $t = 0.38$ just before that collapse.

In the next figures we plot evolutions of initial "∞-like" curve. In Figures 3a

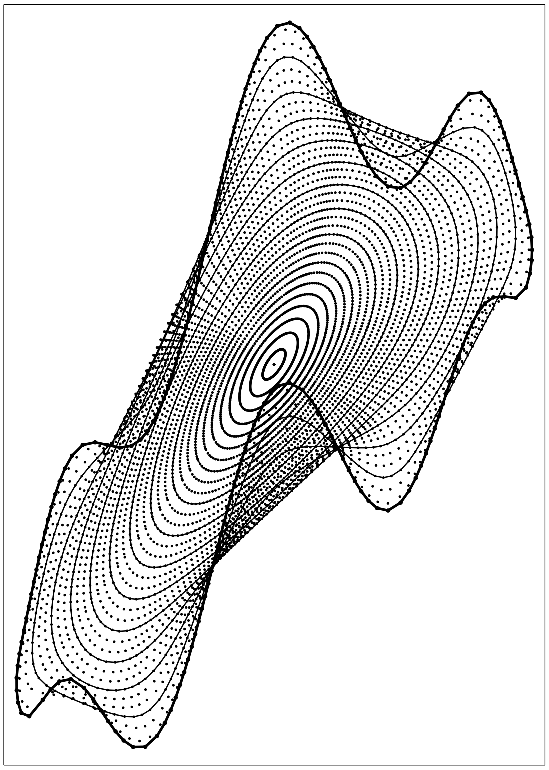

Fig. 2a. $\beta(k) = k^{1/3}$, discrete evolution using tangential redistribution of grid points preserving the relative local length.

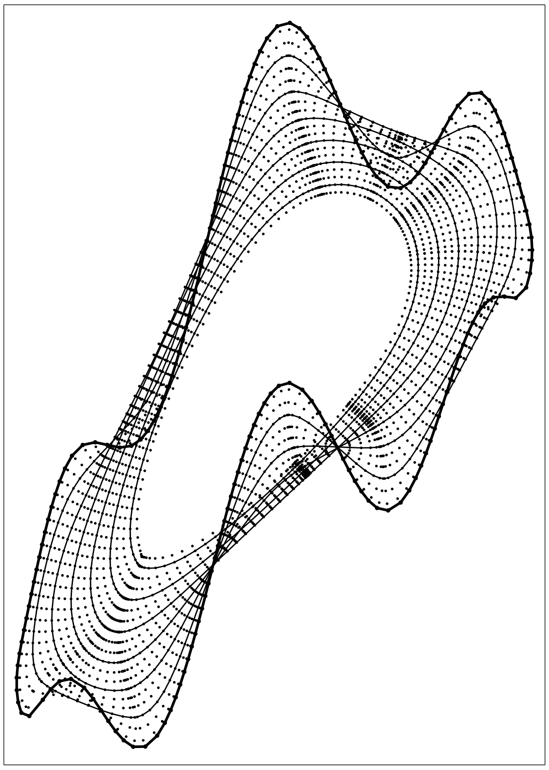

Fig. 2b. $\beta(k) = k^{1/3}$, without redistribution, computation collapses due to vanishing of the local length element $|\partial_u x|$.

and Figure 3c, the tangential velocity preserving relative local length has been used whereas in Figure 3b one sees that the computation without tangential redistribution cannot prevent vanishing of $|\partial_u x|$. In Figures 4a-4b, the evolutions of general nonconvex curves are plotted.

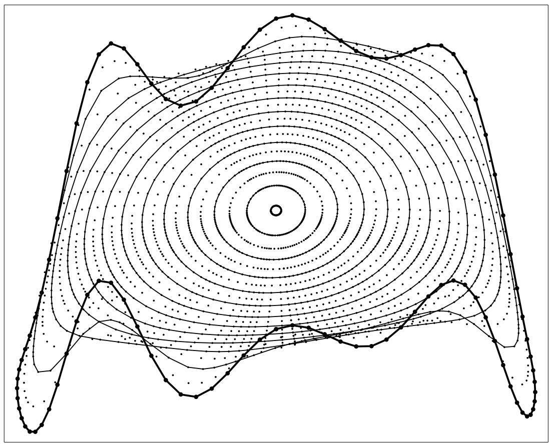

Fig. 4a. $\beta(k) = k^{1/2}$, evolution of general nonconvex curve using tangential redistribution of points.

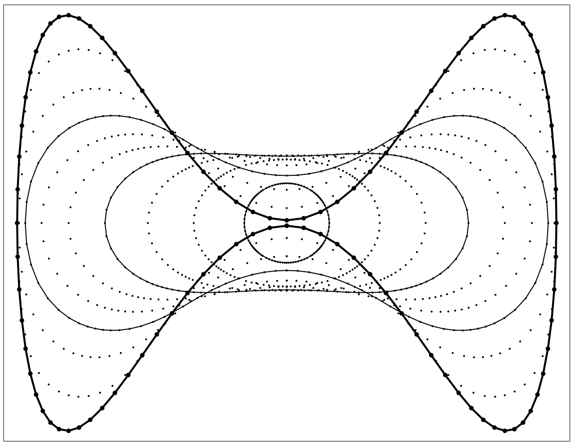

Fig. 4b. $\beta(k) = k^{2}$, evolution of general nonconvex curve using tangential redistribution of points.

In Figures 5a - 5b the affine invariant evolution of initial ellipse with half-axes

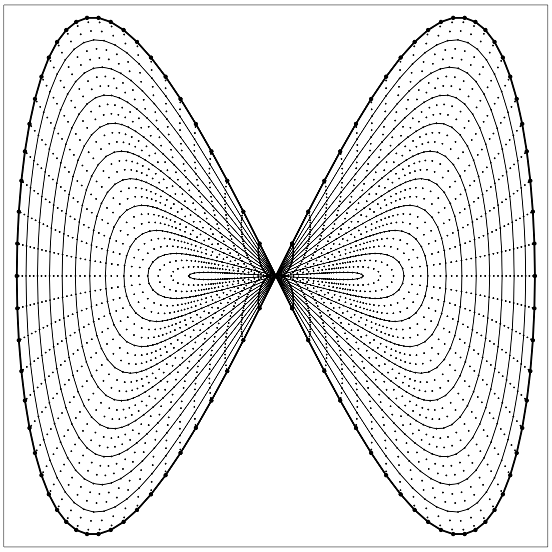

Fig. 3a. $\beta(k) = k^{1/4}$, evolution of "∞-like" curve using redistribution.

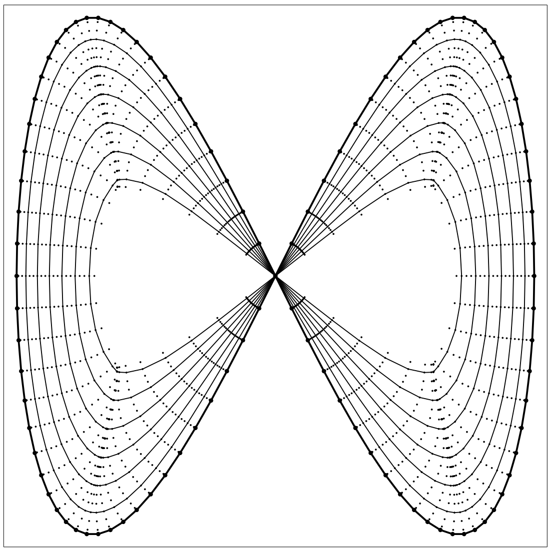

Fig. 3b. $\beta(k) = k^{1/4}$, evolution of "∞-like" curve without redistribution leading to merging of points.

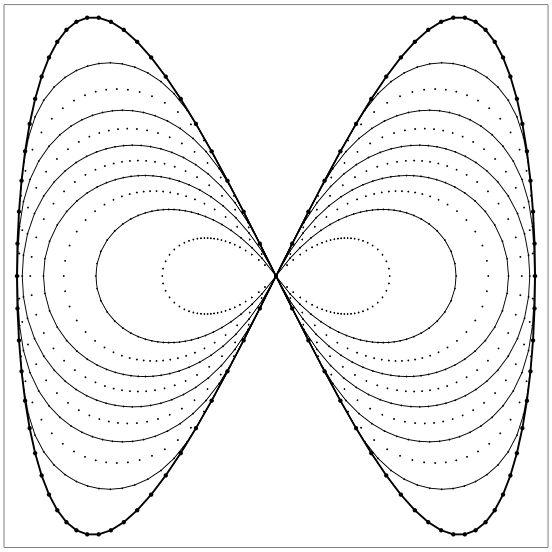

Fig. 3c. $\beta(k) = k^2$, evolution of "∞-like" curve using tangential redistribution of points.

ratio 3:1 is documented. In Figure 5a, the exact blow up time $T_{max} = 1.560$ while the numerically computed one is equal 1.570 using time step $\tau = 0.001$ and 100 grid points for curve representation. The half-axes ratio as well as isoperimetric ratio were perfectly conserved during numerical evolution. Without a tangential velocity (i.e. $\alpha = 0$ and $\theta_2 = 1$), the numerical solution collapses as one can see in Figure 5b.

In Figures 6a - 7c we present various computations including anisotropy in the model. For Figures 6a - 6d we have chosen three-fold anisotropy while for Figures 7a - 7b four-fold one. In Figure 6a, we have computed linear anisotropic evolution

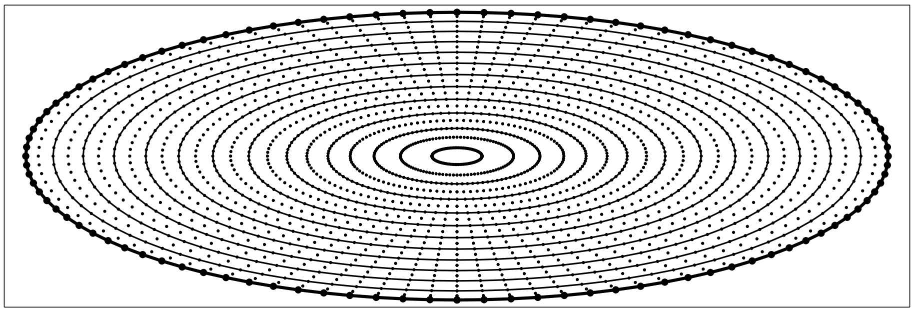

Fig. 5a. Affine invariant motion of ellipse using tangential redistribution of points.

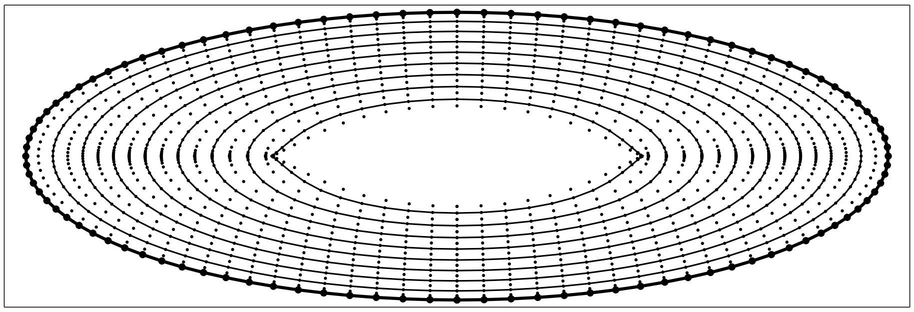

Fig. 5b. Computation using only normal component of velocity.

of unit circle by algorithm (7.1)-(7.3), in Figures 6b - 6c we combine anisotropy with a nonlinear function of curvature. Figure 6d represents the same parameters of the model as in Figure 6c but without point redistribution which causes collapse of computation. The curves are represented by 100 grid points and $\tau = 0.001$. In the first numerical experiment documented in Figure 6a, the numerical blow-up time $T_{max} = 0.509$ (exact one is 0.5), the isoperimetric ratio tends to 1.048 representing the Wulf shape for such a anisotropy function. In Figure 6b we chose $\beta(k) = k^m, m > 1$, the evolution is faster, numerical $T_{max} = 0.373$, $(m = 2)$ and asymptotic isoperimetric ratio is 1.014. Taking $\beta(k) = k^m, m < 1$ the anisotropic evolution is slowed down, numerical $T_{max} = 0.601$, $m = \frac{1}{2}$, isoperimetric ratio tends to 1.13, asymptotical shape is more "sharp". In this example one sees that the initial uniform redistribution of grid points is not kept perfectly (in spite of results in Figure 6b). Very likely it is caused by not well-conditioning of linear systems. This phenomenon is an objective of our future study. Further anisotropic experiments are presented in Figures 7a-7c, where a convergence to "oval square" corresponding to anisotropy is observed in both convex and nonconvex cases. The evolution of nonconvex curve from Figure 7c is computed also using three-fold anisotropy and results are plotted in Figure 7d. The last numerical experiment represents affine invariant evolution of spiral. In Figure 8 we document several time moments of the motion until it is shrinking to a point.

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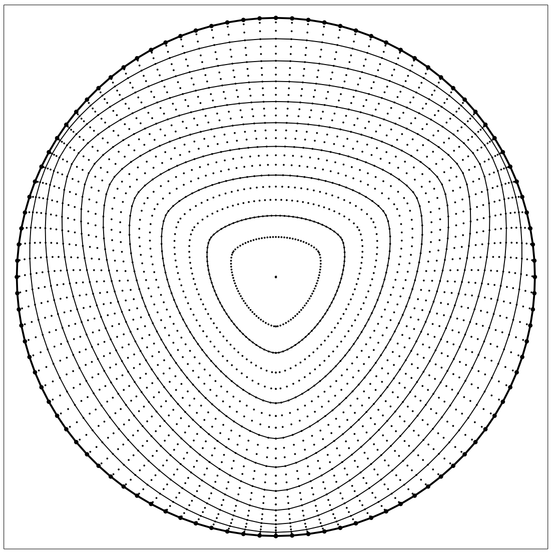

Fig. 6a. $\beta(k,\nu) = (1 - 7/9\cos(3\nu))k$, using redistribution.

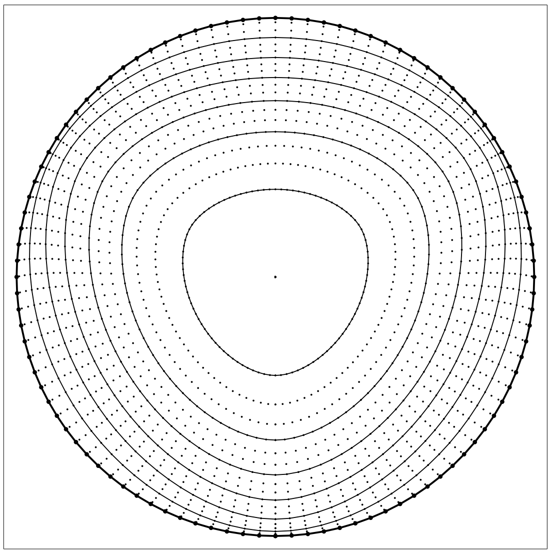

Fig. 6b. $\beta(k,\nu) = (1 - 7/9\cos(3\nu))k^2$, using redistribution.

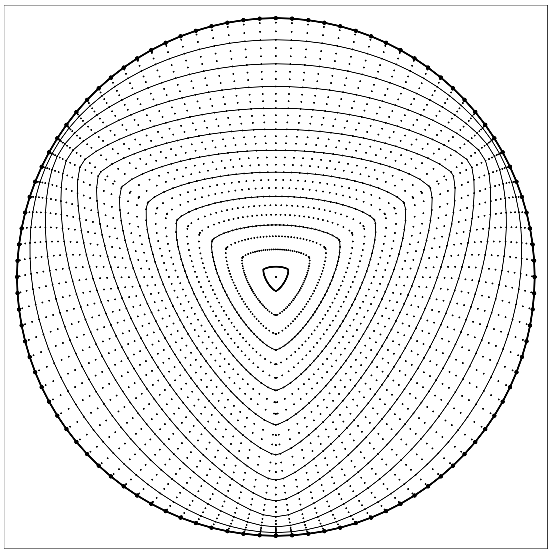

Fig. 6c. $\beta(k,\nu) = (1 - 7/9\cos(3\nu))k^{1/2}$, using redistribution.

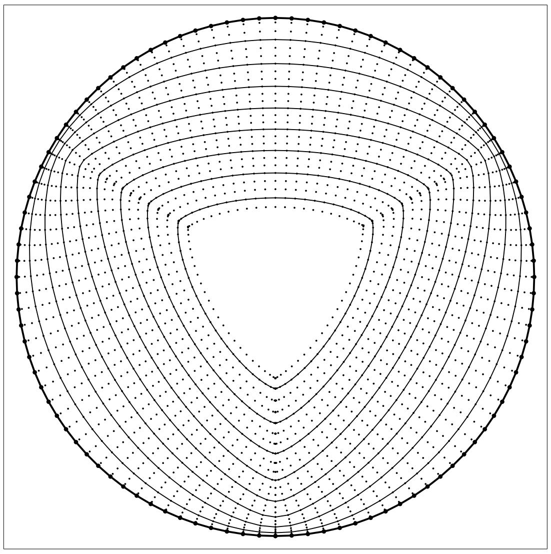

Fig. 6d. $\beta(k,\nu) = (1 - 7/9\cos(3\nu))k^{1/2}$, without redistribution.

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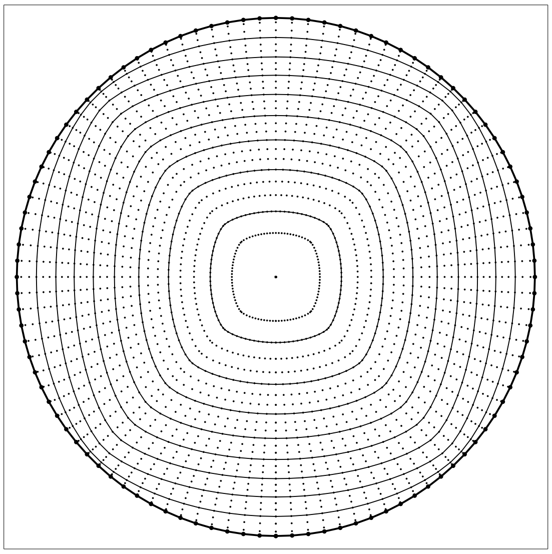


Fig. 7a. $\beta(k,\nu) = (1 - 0.8\cos(4(\nu - \pi/4)))k$, using redistribution

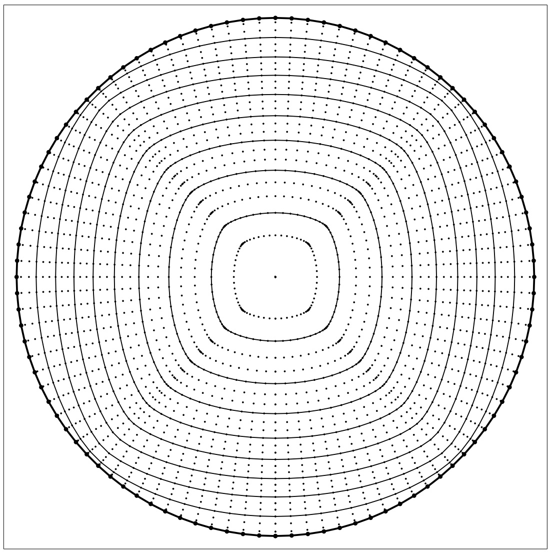


Fig. 7b. $\beta(k,\nu) = (1 - 0.8\cos(4(\nu - \pi/4)))k$, without redistribution

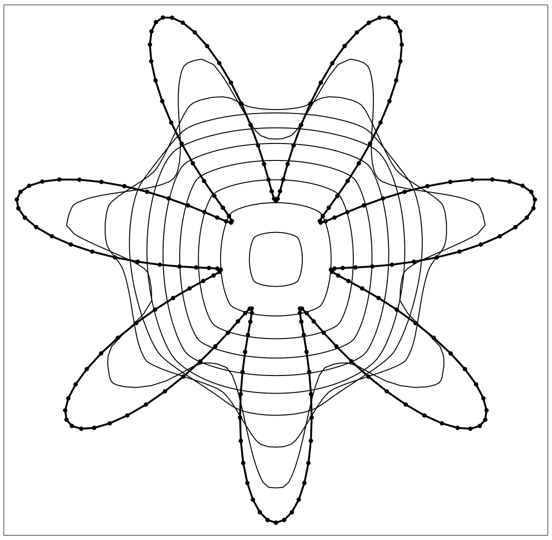


Fig. 7c. $\beta(k,\nu) = (1 - 0.8\cos(4(\nu - \pi/4)))k^{3/4}$ using redistribution

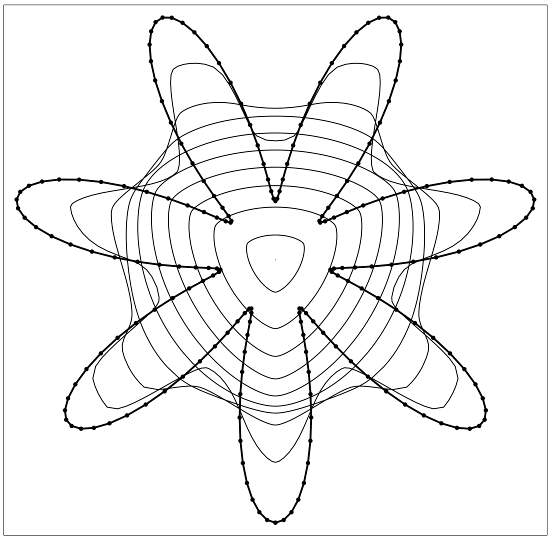
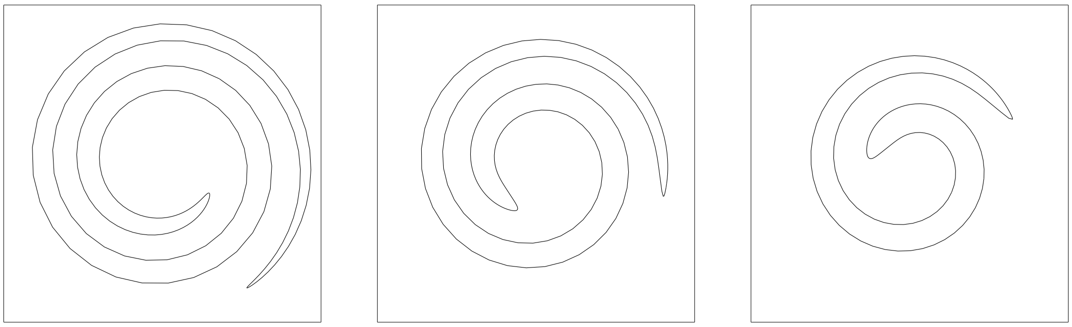


Fig. 7d. $\beta(k,\nu) = (1 - 7/9\cos(3\nu))k^{3/4}$ using redistribution



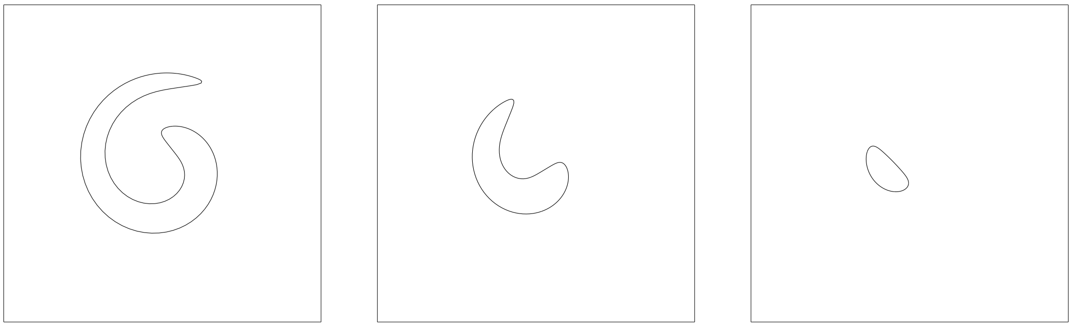


Fig. 8. The sequence of evolving spirals for $\beta(k,\nu) = k^{1/3}$ using redistribution. The limiting curve is an ellipse rounded point.

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K. Mikula, Department of Mathematics, Slovak University of Technology, Radlinského 11, 813 68 Bratislava, Slovak Republic, *e-mail mikula@vox.svf.stuba.sk*

Daniel Ševčovič, Inst. of Applied Mathematics, Faculty of Math.& Physics, Comenius University, 842 15 Bratislava, Slovak Republic, *e-mail sevcovic@fmph.uniba.sk*