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On an iterative approach to solving the nonlinear satellite-fixed geodetic boundary-value problem

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Abstract The paper deals with an iterative treatment of solving the nonlinear satellite-fixed geodetic boundary-value problem (NSFGBVP). To that goal we formulate the NSFGBVP consisting of the Laplace equation in 3D bounded domain outside the Earth. The computational domain is bounded by the approximation of the Earth's surface where the nonlinear boundary condition (BC) with prescribed magnitude of the gravity vector is given and by a spherical boundary placed approximately at the altitude of chosen satellite mission on which the Dirichlet BC for disturbing potential obtained from the satellite only geopotential model is applied. In case of local gravity field modelling, we add another four side boundaries where the Dirichlet BC is prescribed as well. The concept of our iterative approach is based on determining the direction of actual gravity vector together with the value of the disturbing potential. Such an iterative approach leads to the first iteration where the classical fixed gravimetric boundary-value problem with the oblique derivative BC is solved and the last iteration represents the approximation of the actual disturbing potential and the direction of gravity vector. As a numerical method for our approach, the finite volume method has been implemented. The practical numerical experiments deal with the local and global gravity field modelling. In case of local gravity field modelling, namely in the domain above Slovakia, the disturbing potential as a direct numerical result is transformed to the quasigeoidal heights and tested by the GPS-levelling. Results show an improvement in the standard deviation for subsequent iterations in solving NSFGBVP as well as the convergence to EGM2008. The differences between the last and

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the first iteration, which represent the numerically obtained linearization error, reach up to $10\,cm$. In case of global gravity field modelling, our solution is compared with the disturbing potential generated from EGM2008. The obtained numerical results show that the error of the linearization can exceed several centimeters, mainly in high mountainous areas (e.g. in Himalaya region they reach $20\,cm$) as well as in areas along the ocean trenches (varying from $-2.5\,cm$ to $2.5\,cm$).

Keywords Nonlinear boundary value problem · Finite volume method · Iterative approach

1 Formulation of the nonlinear satellite-fixed geodetic boundary-value problem

The nonlinear geodetic boundary-value problem (BVP) has been of interest of many scientists and researches. A uniqueness theorem for the fixed gravimetric BVP (FGBVP) was first given by Backus (1968). Later Koch and Pope (1972) presented a uniqueness proof for the nonlinear geodetic BVP. The free nonlinear BVP exactly solved by metric continuation was discussed by Graferend and Niemeier (1971) as well as by Graferend et al. (1989). Then Bjerhammar and Svensson (1983) used the general implicit function theorem and gave a solution of the existence and uniqueness problem in the nonlinear case. Expanding the nonlinear boundary condition into a Taylor series, based upon some reference potential field approximating the geopotential, was shown by Heck (1989). Sacerdote and Sansó (1989) further developed the idea used by Bjerhammar and Svensson for an iterative solution and they found explicit convergence conditions. They calculated the respective constant governing the convergence in the ideal case of a spherical boundary. Finally, we should mention authors Georgio Díaz, Jesús Díaz and Otero who showed the existence and uniqueness of a viscosity solution for the Backus problem (Díaz et al., 2006; Diaz et al., 2011).

Let us consider the non-homogeneous elliptic equation of second order outside the Earth

$$\Delta W(\mathbf{x}) = 2\omega^2, \tag{1}$$

where $W(\mathbf{x})$ is the actual gravity potential and ω is the spin velocity of the Earth. The norm of gradient of the gravity potential $W(\mathbf{x})$ is

$$|\nabla W(\mathbf{x})| = g(\mathbf{x}), \tag{2}$$

where $g(\mathbf{x})$ denotes the magnitude of so-called total gravity vector. When $g(\mathbf{x})$ is prescribed on the Earth's surface, Eq. (1) with BC (2) represents the nonlinear geodetic BVP for the actual gravity potential $W(\mathbf{x})$.

The actual gravity field can be expressed as a sum of the selected model field and the remainder of the actual field (Hofmann-Wellenhof and Moritz, 2005), for corresponding potentials we can write

$$W(\mathbf{x}) = U(\mathbf{x}) + T(\mathbf{x}), \tag{3}$$

where $U(\mathbf{x})$ is the normal gravity potential and $T(\mathbf{x})$ the disturbing potential. When the model field is generated by a massive ellipsoid rotating with the Earth with the same spin velocity ω, its constant surface potential is equal to geopotential W_0 and its mass is the same as the mass of the Earth, then the disturbing potential $T(\mathbf{x})$ outside the Earth will satisfy the Laplace equation $\Delta T(\mathbf{x}) = 0$. It follows from the fact that $T(\mathbf{x})$ does not have any centrifugal component since the centrifugal component of the Earth is the same as the centrifugal component of the chosen model.

Now let us consider the bounded domain Ω depicted in Fig. 1. Such a domain is set in the external space above the Earth where the bottom surface $\Gamma \subset \partial\Omega$, where $\partial\Omega$ denotes a boundary of Ω, represents a part of the Earth's surface and the upper part of the boundary is at altitude of the chosen satellite mission. On the lower part of the boundary the nonlinear BC coming from (2) is given. On the upper spherical part of the domain as well as on the side boundaries, the Dirichlet-type BC (Eymard et al., 2001) obtained from satellite gravity missions is prescribed. That allows us to fix our solution to the satellite data. It is worth noting that another BC (Neumann or Newton BC) derived from satellite gravity missions suitable for the elliptic equation of second order might be taken into account as well.

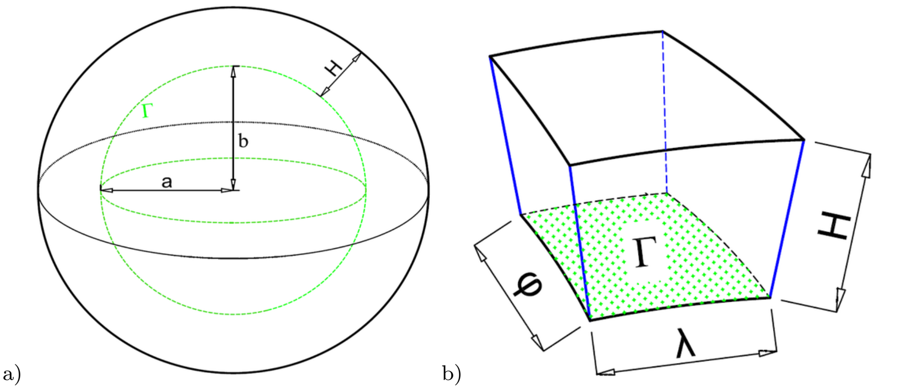

Fig. 1 Sketch of the computational domain Ω for a) global numerical experiment, b) local numerical experiment. The dotted boundary Γ represents the part of the Earth's surface, φ and λ denote latitude and longitude and H denotes the height above WGS84.

Then our nonlinear satellite-fixed geodetic BVP (NSFGBVP) for the disturbing potential $T(\mathbf{x})$ is formulated in the following form

$$\Delta T(\mathbf{x}) = 0 \quad \mathbf{x} \in \Omega, \tag{4}$$

$$|\nabla(T(\mathbf{x}) + U(\mathbf{x}))| = g(\mathbf{x}) \quad \mathbf{x} \in \Gamma, \tag{5}$$

$$T(\mathbf{x}) = T_{SAT}(\mathbf{x}) \quad \mathbf{x} \in \partial\Omega - \Gamma. \tag{6}$$

where T_{SAT} is the disturbing potential generated from a chosen satellite only model based on the spherical harmonics. It is worth to note that we are looking for a solution in a bounded domain Ω, so we do not deal with its regularity at infinity. The influence of BC applied on side boundaries has been studied by Fašková et al. (2010).

In general, one can write the norm of the gradient of the gravity potential in the form

$$|\nabla W(\mathbf{x})| = \frac{\nabla W(\mathbf{x})}{|\nabla W(\mathbf{x})|} \cdot \nabla W(\mathbf{x}). \tag{7}$$

By inserting (7) in equation (5), we obtain

$$\frac{\nabla(T(\mathbf{x}) + U(\mathbf{x}))}{|\nabla(T(\mathbf{x}) + U(\mathbf{x}))|} \cdot \nabla(T(\mathbf{x}) + U(\mathbf{x})) = g(\mathbf{x}) \tag{8}$$

and if we denote

$$\mathbf{v}(\mathbf{x}) = \frac{\nabla(T(\mathbf{x}) + U(\mathbf{x}))}{|\nabla(T(\mathbf{x}) + U(\mathbf{x}))|}, \tag{9}$$

we can rewrite the BC (5) as

$$\mathbf{v}(\mathbf{x}) \cdot \nabla(T(\mathbf{x})) = g(\mathbf{x}) - \mathbf{v}(\mathbf{x}) \cdot \nabla(U(\mathbf{x})) \quad \mathbf{x} \in \Gamma. \tag{10}$$

Since the unit vector $\mathbf{v}(\mathbf{x})$, defining the direction of the actual gravity vector, is unknown and depends on $T(\mathbf{x})$, BC (10) is still nonlinear, but its form allows to use an iterative approach for determining $\mathbf{v}(\mathbf{x})$ and $T(\mathbf{x})$ such that (4)-(6) is fulfilled. The iterative procedure for solving NSFGBVP will be defined as follows

$$\Delta T^{n+1}(\mathbf{x}) = 0 \quad \mathbf{x} \in \Omega, \tag{11}$$

$$\mathbf{v}^n(\mathbf{x}) \cdot \nabla(T^{n+1}(\mathbf{x})) = g(\mathbf{x}) - \mathbf{v}^n(\mathbf{x}) \cdot \nabla(U(\mathbf{x})) \quad \mathbf{x} \in \Gamma, \tag{12}$$

$$T^{n+1}(\mathbf{x}) = T_{SAT}(\mathbf{x}) \quad \mathbf{x} \in \partial\Omega - \Gamma, \tag{13}$$

for $n = 0, 1, 2, \ldots$, where

$$\mathbf{v}^n(\mathbf{x}) = \frac{\nabla(T^n(\mathbf{x}) + U(\mathbf{x}))}{|\nabla(T^n(\mathbf{x}) + U(\mathbf{x}))|}, \tag{14}$$

and we start the iterations by choosing $T^0(\mathbf{x}) = 0$, i.e. $W^0(\mathbf{x}) = U(\mathbf{x})$ and correspondingly for $\mathbf{v}^0(\mathbf{x})$ we get

$$\mathbf{v}^0(\mathbf{x}) = \frac{\nabla(U(\mathbf{x}))}{|\nabla(U(\mathbf{x}))|} = \mathbf{s}(\mathbf{x}), \tag{15}$$

where $\mathbf{s}(\mathbf{x})$ represents the direction of the normal gravity vector. One can see that in every iteration we solve the geodetic BVP for $T^{n+1}(\mathbf{x})$ with prescribed oblique derivative vector $\mathbf{v}^n(\mathbf{x})$. In the first step we solve the linearized fixed gravimetric BVP (FGBVP) (Koch and Pope, 1972; Holota, 1997, 2005; Čunderlík et al., 2008; Fašková et al., 2010) with the oblique derivative given by

$$\mathbf{s}(\mathbf{x}) \cdot \nabla(T^1(\mathbf{x})) = g(\mathbf{x}) - \gamma(\mathbf{x}) = \delta g(\mathbf{x}), \tag{16}$$

where $\gamma(\mathbf{x}) = |\nabla(U(\mathbf{x}))|$ and denotes a magnitude of the normal gravity vector and $\delta g(\mathbf{x})$ denotes the gravity disturbance. In further iterations we improve the direction of the unit vector $\mathbf{v}(\mathbf{x})$. Such a process reduces the linearization error. Since we solve the problem iteratively, we need a stopping criterion. To that goal we use a difference of two successive iterations and stop the procedure, if in each point the inequality

$$|T^n(\mathbf{x}) - T^{n+1}(\mathbf{x})| < \varepsilon, \tag{17}$$

holds, where ε means a user-specified small real number. The last iteration represents our approximation of the disturbing potential $T(\mathbf{x})$ and direction of gravity vector $\mathbf{v}(\mathbf{x})$ in (4) - (6), and the sum $T^{n+1}(\mathbf{x}) + U(\mathbf{x})$ represents the approximation of actual gravity potential $W^{n+1}(\mathbf{x})$ in every point of the computational domain Ω.

2 Numerical solution of the nonlinear satellite-fixed geodetic boundary-value problem

We can see that in each step of our iterative process (11)-(13) we deal with the oblique derivative BVP defined as

$$\Delta T(\mathbf{x}) = 0 \quad \mathbf{x} \in \Omega, \tag{18}$$

$$\mathbf{v}(\mathbf{x}) \cdot \nabla(T(\mathbf{x})) = g(\mathbf{x}) - \mathbf{v}(\mathbf{x}) \cdot \nabla(U(\mathbf{x})) = \alpha(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \tag{19}$$

$$T(\mathbf{x}) = T_{SAT}(\mathbf{x}) \quad \mathbf{x} \in \partial\Omega - \Gamma. \tag{20}$$

To solve (18)-(20), we have chosen the finite volume method (FVM), (Eymard et al., 2001). In FVM we divide the computational domain Ω into finite volumes p, multiply the Laplace equation by minus one and integrate the resulting equation over each finite volume with a use of the divergence theorem that turns the volume integral into the surface integral,

$$-\int_{p} \Delta T \; dxdydz = -\int_{\partial p} \nabla T . \mathbf{n} \; d\sigma, \tag{21}$$

from where we get the *weak formulation* of the equation (18) in the finite volume p

$$-\int_{\partial p} \frac{\partial T}{\partial n} d\sigma = 0. \tag{22}$$

Let $q \in N(p)$ be a neighbour of the finite volume p, where $N(p)$ denotes all neighbours of p. Let T_p and T_q be approximate values of T in p and q, e_{pq} be a boundary of the finite volume p common with q, $\mathbf{n}_{pq}$ be its unit normal vector oriented from p to q, $m(e_{pq})$ is the area of e_{pq}. Let x_p and x_q be representative points of p and q respectively and d_{pq} their distance. If we approximate the normal derivative along the boundary of the finite volume p by

$$\frac{\partial T}{\partial n_{pq}} \approx \frac{T_q - T_p}{d_{pq}}, \tag{23}$$

we obtain from (22) and (23) the following equation for every finite volume p

$$\sum_{q\in N(p)} \frac{m(e_{pq})}{d_{pq}}(T_p - T_q) = 0, \tag{24}$$

which forms together the linear system of algebraic equations. The term $\frac{m(e_{pq})}{d_{pq}}$ defined on sides of the finite volume p is referred to as the transmissivity coefficient (Eymard et al., 2001). Then we define indices $i = 1, ..., n_1$, $j = 1, ..., n_2$ and $k = 1, ..., n_3$ in the direction of the longitude λ, latitude φ and height h, where n_1, n_2 and n_3 denote the numbers of dicretisation intervals in zonal, meridional and height's direction, respectively. In this way we obtain the linear system of equations that can be written in the form

$$\begin{aligned} P_{i,j,k}T_{i,j,k} - W_{i,j,k}T_{i-1,j,k} - E_{i,j,k}T_{i+1,j,k} - N_{i,j,k}T_{i,j+1,k} - \\ -S_{i,j,k}T_{i,j-1,k} - U_{i,j,k}T_{i,j,k+1} - D_{i,j,k}T_{i,j,k-1} = 0, \end{aligned} \tag{25}$$

where $P_{i,j,k}, W_{i,j,k}, E_{i,j,k}, N_{i,j,k}, S_{i,j,k}, U_{i,j,k}$ and $D_{i,j,k}$ are transmissivity coefficients and their derivation can be found in Macák et al. (2012).

The system (25) must be accompanied by the boundary conditions. In case of the Dirichlet BC, we prescribe the value of T_q on the boundary, while in case of the oblique derivative BC, a special treatment is needed. For the bottom boundary, when $k = 1$, we add new finite volumes p signed by index $k = 0$. Then we split the gradient of $T(\mathbf{x})$ in (19) into one normal and two tangential directions

$$\nabla\mathbf{T} = (\nabla T.\mathbf{n})\mathbf{n} + (\nabla T.\mathbf{t_1})\mathbf{t_1} + (\nabla T.\mathbf{t_2})\mathbf{t_2} = \frac{\partial T}{\partial n}\mathbf{n} + \frac{\partial T}{\partial t_1}\mathbf{t_1} + \frac{\partial T}{\partial t_2}\mathbf{t_2}, \tag{26}$$

where $\mathbf{n}$ is the unit normal vector and $\mathbf{t_1}$, $\mathbf{t_2}$ are linearly independent unit tangent vectors to $\Gamma \subset \partial\Omega \subset R^3$. So the BC (19) is transformed into the form

$$\frac{\partial T}{\partial n}(\mathbf{n}.\mathbf{v}) + \frac{\partial T}{\partial t_1}(\mathbf{t_1}.\mathbf{v}) + \frac{\partial T}{\partial t_2}(\mathbf{t_2}.\mathbf{v}) = \alpha. \tag{27}$$

Then we approximate the normal and tangential derivatives according to notations depicted in Fig. 2

$$\frac{\partial T}{\partial n} \approx \frac{T_D - T_P}{|\mathbf{x}_D - \mathbf{x}_P|}, \quad \frac{\partial T}{\partial t_1} \approx \frac{T_{EN} - T_{WS}}{|\mathbf{x}_{EN} - \mathbf{x}_{WS}|}, \quad \frac{\partial T}{\partial t_2} \approx \frac{T_{WN} - T_{ES}}{|\mathbf{x}_{WN} - \mathbf{x}_{ES}|},$$

where we have denoted values $T_{i,j,k-1}$ and $T_{i,j,k}$ by T_D and T_P, respectively. Values T_{EN}, T_{WS}, T_{WN}, T_{ES} are obtained as follows

$$T_{WN} = \frac{T_{i,j,k} + T_{i,j-1,k} + T_{i,j,k-1} + T_{i,j-1,k-1} + T_{i-1,j,k} + T_{i-1,j-1,k} + T_{i-1,j,k-1} + T_{i-1,j-1,k-1}}{8},$$

$$T_{EN} = \frac{T_{i,j,k} + T_{i,j-1,k} + T_{i,j,k+1} + T_{i,j-1,k+1} + T_{i-1,j,k} + T_{i-1,j-1,k} + T_{i-1,j,k+1} + T_{i-1,j-1,k+1}}{8},$$

$$T_{WS} = \frac{T_{i,j,k} + T_{i,j+1,k} + T_{i,j,k-1} + T_{i,j+1,k-1} + T_{i-1,j,k} + T_{i-1,j+1,k} + T_{i-1,j,k-1} + T_{i-1,j+1,k-1}}{8},$$

$$T_{ES} = \frac{T_{i,j,k} + T_{i,j+1,k} + T_{i,j,k+1} + T_{i,j+1,k+1} + T_{i-1,j,k} + T_{i-1,j+1,k} + T_{i-1,j,k+1} + T_{i-1,j+1,k+1}}{8},$$

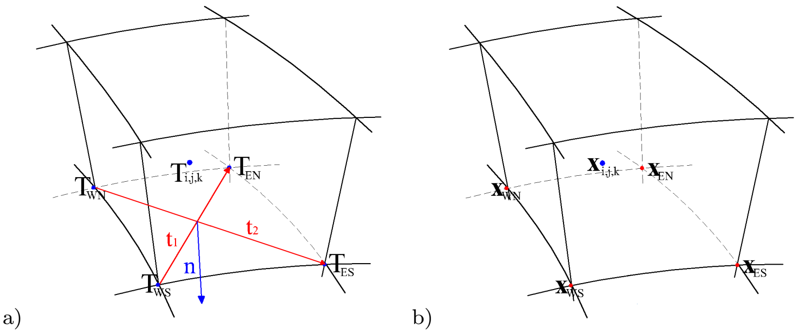

Fig. 2 Brief illustration of the computational grid for an approximation of the oblique derivative. a) T_{ijk} denotes the value of the disturbing potential in the center of volume. $T_{WS}, T_{ES}, T_{EN}, T_{WN}$ are values of the disturbing potential in the vertices. Vectors $\mathbf{t_1}$ and $\mathbf{t_2}$ denote independent tangent vectors to Γ and $\mathbf{n}$ the normal vector to Γ. b) $\mathbf{x}_{ijk}$ denotes position vector of the center of volume and $\mathbf{x}_{WS}, \mathbf{x}_{ES}, \mathbf{x}_{EN}, \mathbf{x}_{WN}$ are values of the position vectors of the vertices.

and $\mathbf{x}_D, \mathbf{x}_P, \mathbf{x}_{EN}, \mathbf{x}_{WS}, \mathbf{x}_{WN}, \mathbf{x}_{ES}$ are their corresponding position vectors, see Fig. 2. More details can be found in Macák et al. (2012). Then the final discrete form of the oblique derivative BC is given by

$$\mathbf{v} \cdot \nabla(T(\mathbf{x})) \approx \frac{T_D - T_P}{|\mathbf{x}_D - \mathbf{x}_P|}(\mathbf{n}.\mathbf{v}) + \frac{T_{EN} - T_{WS}}{|\mathbf{x}_{EN} - \mathbf{x}_{WS}|}(\mathbf{t_1}.\mathbf{v}) + \\ + \frac{T_{WN} - T_{ES}}{|\mathbf{x}_{WN} - \mathbf{x}_{ES}|}(\mathbf{t_2}.\mathbf{v}) = \alpha.$$

3 Numerical experiments

The local numerical experiment was performed in the domain above Slovakia bounded by $\varphi \in \langle 47.0°, 50.5° \rangle$ and $\lambda \in \langle 16.0°, 23.0° \rangle$. The bottom boundary was created using heights generated from SRTM30 PLUS (Becker et al., 2009) and the upper boundary was at the height of 240 km above WGS84, corresponding to an average altitude of the satellite orbit. The number of finite volumes was 1000 in height, 630 in meridional and 840 in zonal directions, i.e. the resolution with respect to latitude and longitude was 30" $\times$20". We started our computations by solving the linearized FGBVP where the surface gravity disturbances were applied on the bottom boundary Γ. They were generated from an available dataset of terrestrial gravity data in Slovakia (Grand et al., 2001) while ellipsoidal heights of gravimetric measurements were computed from levelling heights using EGM2008 (Pavlis et al., 2012). On the upper and side boundaries, the disturbing potential generated from the GOCO03s satellite-only model (Mayer-Gürr et al., 2012) was prescribed. Computations were performed on 30 processors using 78 GB of distributed memory taking approximately 5 hours of total CPU time per processor. To reach the prescribed stopping criterium $\varepsilon = 10^{-3}[m^2s^{-2}]$, 10 iterations were needed. Results are

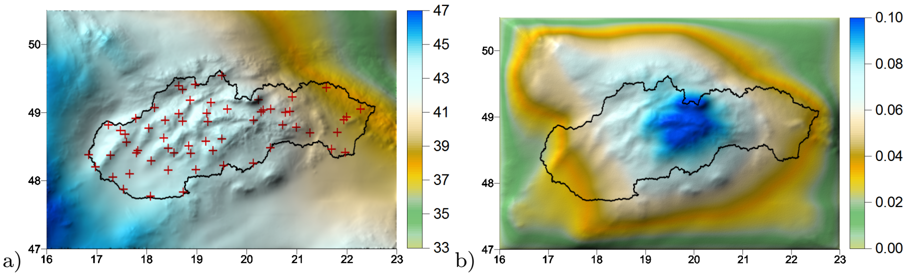

Fig. 3 a) Quasigeoidal heights $\zeta[m]$ obtained by solving the NSFGBVP, b) Differences in $\zeta[m]$ between 10^{th} and 1^{st} iteration. Red crosses denote the distribution of 61 GPS/leveling points.

Table 1 Statistics of residuals [m] between our NSFGBVP solution and quasigeoidal heights obtained by GPS/levelling at 61 points in the area of Slovakia.

	1^{st} iter.	5^{th} iter.	8^{th} iter.	10^{th} iter.	EGM2008
Min. value	0.151	0.209	0.229	0.248	0.301
Mean value	0.284	0.325	0.348	0.352	0.437
Max. value	0.422	0.459	0.476	0.493	0.584
St. deviation	0.055	0.049	0.047	0.046	0.043

presented in Table 1 and Fig. 3. One can observe an improvement in the standard deviation for subsequent iterations in solving NSFGBVP (Tab. 1) as well as the convergence to EGM2008. The differences between the 10^{th} and 1^{st} iteration, which represent the numerically obtained linearization error, reach up to $10\,cm$.

The global numerical experiment dealt with the high-resolution global gravity field modelling in the computational domain Ω bounded by the bottom boundary approximating the real Earth's surface created by using heights generated from SRTM30 PLUS and by a surface at height of 240 km above WGS84 corresponding to the average altitude of satellite orbit. The number of divisions was $4320 \times 2160 \times 600$ leading to the resolution $5' \times 5' \times 400\,m$. Again we start with the linearized FGBVP consisting of gravity disturbances interpolated from the DTU10-GRAV gravity field model (Andersen, 2010) and applied on the bottom boundary. On the upper boundary the disturbing potential generated from GOCO03s was prescribed. The stopping criterium was $\epsilon = 10^{-3}[m^2 s^{-2}]$ and again, 10 iterations were needed. The FVM solutions obtained in each iteration are compared with EGM2008. Statistical characteristics of residuals are presented in Table 2. Figure 4 depicts differences between the 10^{th} and 1^{st} iteration. They represent the numerically obtained linearization error in the linearized FGBVP. One can observe that our iteration approach improves solution mainly in areas of high mountains (e.g. in Himalaya region they reach $20\,cm$) as well as in areas along the ocean trenches (varying from $-2.5\,cm$ to $2.5\,cm$).

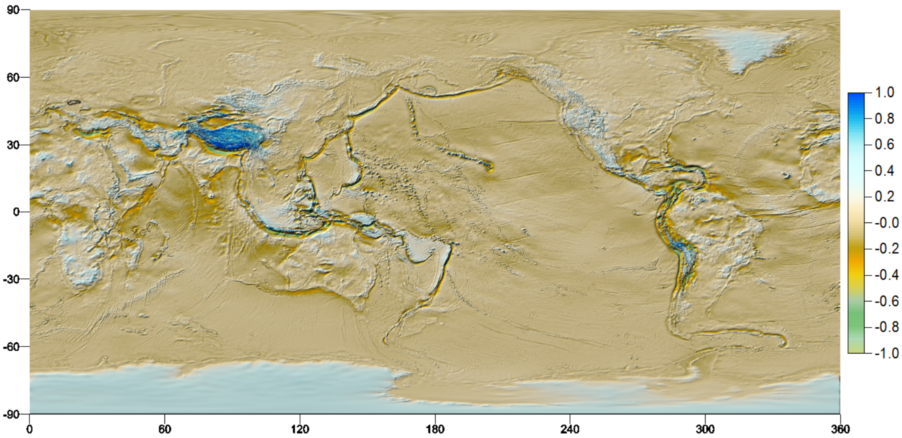

Fig. 4 Differences in $T[m^2s^{-2}]$ between 10^{th} and 1^{st} iteration, representing the numerically obtained linearization error.

Table 2 Statistics of residuals $[m^2s^{-2}]$ between the disturbing potential obtained by solving NSFGBVP and the disturbing potential generated from EGM2008 in the global experiment.

	Min. value		Mean value		Max. value		St. dev.	
Iter.	1^{st}	10^{th}	1^{st}	10^{th}	1^{st}	10^{th}	1^{st}	10^{th}
TOTAL	-2.150	-1.985	0.004	0.001	6.143	4.158	0.501	0.419
SEA	-0.705	-0.632	-0.021	-0.011	1.131	1.019	0.206	0.199
LAND	-2.150	-1.985	0.035	0.029	6.143	4.158	0.855	0.768

4 Summary and conclusions

We have presented an iterative approach to solving the nonlinear satellite-fixed geodetic boundary-value problem (NSFGBVP) defined in this paper. The NSFGBVP has been solved by the finite volume method, where the direction of the actual gravity vector as well as the disturbing potential are updated in each iteration. In the first iteration, the linearized FGBVP is solved together with the oblique derivative problem. Next iterations treat its numerically obtained linearization error. The obtained numerical results show that the error of the linearization can exceed several centimeters, mainly in high mountainous areas and along ocean trenches. This indicates that for precise gravity field modeling it is necessary to deal with the nonlinear geodetic BVPs avoiding the linearization error. Presented numerical experiments show that the proposed iterative approach converges while the study of its convergence from theoretical point of view will be a task of our future research.

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