ELSEVIER

Applied Numerical Mathematics 17 (1995) 47–59

APPLIED NUMERICAL MATHEMATICS

Solution of nonlinear diffusion appearing in image smoothing and edge detection

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Abstract

A numerical approximation of the nonlinear diffusion problem appearing in image processing is discussed. The mathematical model is proposed by Catté, Lions, Morel and Coll and represents an improvement of the original model of Perona and Malik. The scheme is linear, based on Rothe's approximation in time and on the finite element approach in space. The approximating solutions converge strongly in $C(I, L_2) \cap L_2(I, V)$ space to the variational solution.

Keywords: Image smoothing; Edge detection; Nonlinear diffusion equations; Linearization; Full discretization scheme

1. Introduction

In the present paper we are dealing with the numerical approximation of the following nonlinear diffusion problem:

$$\partial_t u - \nabla\big(g(|\nabla G_\sigma * u|)\nabla u\big) = f(u - u_0) \quad \text{in } Q_T \equiv I \times \Omega, \tag{1.1}$$

$$\partial_\nu u = 0 \quad \text{on } I \times \partial\Omega, \tag{1.2}$$

$$u(0, x) = u_0(x), \tag{1.3}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz continuous boundary $\partial\Omega$; $I \equiv (0, T)$ is a

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SSDI 0168-9274(95)00008-9

time interval; $g \in C^{\infty}(\mathbb{R})$, $g(0)=1$ and $g(s) \to 0$ for $s \to \infty$; f is a monotone function; $G_\sigma(x) \in C^{\infty}(\mathbb{R}^2)$ is a smoothing kernel ($\int_{\mathbb{R}^2} G_\sigma(x)\,\mathrm{d}x = 1$ and $G_\sigma(x) \to \delta_x$, the Dirac measure at point x, for $\sigma \to 0$); and $u_\sigma \equiv G_\sigma * u = \int_{\mathbb{R}^2} G_\sigma(x-\xi) u(\xi)\,\mathrm{d}\xi$.

The problem (1.1)–(1.3) arises in the theory of signal filtration, edge detection and image restoration. In this form, it has been proposed and studied in [2,3]. It represents the improvement of the original Perona–Malik model, in which ∇u is used instead of ∇u_σ. The improvement has been given in two main directions: model (1.1)–(1.3) is stable in the presence of noise and is correct from a mathematical point of view (existence and uniqueness of the solution). When $u \in W_2^1(\Omega)$ (Sobolev space) is extended to $\tilde{u}$ on $\mathbb{R}^2$ such that

$$\| \tilde{u} \|_{W_2^1(\mathbb{R}^2)} \leqslant C \| u \|_{W_2^1(\Omega)}$$

then $\nabla G_\sigma * u \to \nabla u$ in $L_2(\Omega)$ for $\sigma \to 0$. Thus, model (1.1)–(1.3) is in a sense close to that of Perona and Malik and keeps all its advantages.

In [3] the Gaussian function

$$G_\sigma(x) = \frac{1}{4\pi\sigma}\, \mathrm{e}^{-|x|^2/4\sigma}$$

has been used as a smoothing kernel.

The initial two-dimensional signal (picture) represented by $u_0(x)$ has to be filtered from a small noise with two opposite requirements—look for the regions of an image where the signal is of constant mean in spite of those regions where the signal changes its tendency. The edges of the signal are indicated by large values of $|\nabla u_\sigma|$ (σ is sufficiently small) and the nonlinear diffusion (1.1)–(1.3) keeps them (due to the shape of the function g the diffusion coefficient is very small on the edges), while for small values of $|\nabla u_\sigma|$ the diffusion coefficient is large; u tends to *const* with time evolution, provided $f \equiv 0$. By means of nondecreasing f ($f(0)=0$) the term $f(u-u_0)$ in (1.1) forces u to be "close" to u_0 and weakens the influence of the stopping time. In [7] it has been proposed to take $f(s)=s$.

The existence and uniqueness of the variational solution of (1.1)–(1.3) has been proved in [3] in the case of $f \equiv 0$. Moreover, the convergence has been proved in $C(I, L_2(\Omega))$ of the approximate solutions u^n, where u^{n+1} is the solution of the linearized parabolic problem

$$\begin{aligned} &\partial_t u^{n+1} - \nabla\big(g(|\nabla G_\sigma * u^n|)\nabla u^{n+1}\big) = 0 \quad \text{in } I \times \Omega, \\ &\partial_\nu u^{n+1} = 0 \quad \text{on } I \times \partial\Omega, \\ &u^{n+1}(0, x) = u_0(x). \end{aligned} \tag{1.4}$$

Our contribution is to prove the convergence in $C(I, L_2(\Omega)) \cap L_2(I, W_2^1(\Omega))$ of the Rothe-type approximation of (1.1)–(1.3) which consists in the following: Let $n \in \mathbb{N}$. We approximate $\partial_t u$ by $\delta u_i := (u_i - u_{i-1})/\tau$ on the time level $t_i = i\tau$, $\tau = T/n$, and for every $i = 1, \dots, n$ let u_i be the solution of linear elliptic equation

$$\delta u_i - \nabla\big(g(|\nabla G_\sigma * u_{i-1}|)\nabla u_i\big) = f(u_{i-1} - u_0), \tag{1.5}$$

with $u_0 = u(0, x)$ from (1.3).

The convergence results concern the Rothe function

$$u^{(n)}(t)=u_{i-1}+(t-t_{i-1})\delta u_i, \qquad \text{for } t_{i-1}\leqslant t\leqslant t_i, \quad i=1,\dots,n, \tag{1.6}$$

which we consider as the approximating solution of the problem (1.1)–(1.3).

The approximation scheme (1.5)–(1.6) is in a sense diagonal according to the iterations (1.4). In practical realizations, the scheme (1.5)–(1.6) has also been used in [3].

Moreover we prove the $C(I, L_2(\Omega))\cap L_2(I, W_2^1(\Omega))$ convergence also for the full discretization scheme, in which (1.5) is projected to finite-dimensional space.

2. Convergence of the approximation scheme (1.5)–(1.6)

Let G_σ be as in Section 1; we shall assume that

$$\operatorname{supp} G_\sigma(x)\subset B_\sigma(0) \tag{2.1}$$

($B_\sigma(0)$ is the ball centered at 0 with radius σ). We can use, e.g.,

$$G_\sigma(x)=\frac{\kappa}{\sigma^2}\exp\left(\frac{|x|^2}{|x|^2-\sigma^2}\right), \qquad \text{for } |x|\leqslant\sigma,$$

$$G_\sigma(x)\equiv 0, \qquad \text{for } |x|>\sigma.$$

$$g \text{ is Lipschitz continuous, } g(0)=1 \text{ and } 0<g(s)\to 0 \text{ for } s\to\infty, \tag{2.2}$$

$$f \text{ is Lipschitz continuous, nondecreasing and } f(0)=0, \tag{2.3}$$

$$u_0\in L_2(\Omega). \tag{2.4}$$

Remark. We use the standard functional spaces $L_2\equiv L_2(\Omega)$, $V\equiv W_2^1(\Omega)$, $L_\infty(\Omega)$, $C(I, L_2)$, V^* (dual to V) and $L_2(I, V^*)$—see, e.g., [5]. We denote the scalar product in $L_2(\Omega)$ by $(\cdot,\cdot)$ and duality between V and V^* by $\langle\cdot,\cdot\rangle$. We denote by $|\cdot|$, $\|\cdot\|$, $\|\cdot\|_*$, $|\cdot|_\infty$, $|\cdot|_{L_2(I,V)}$ the norms in $L_2(\Omega)$, V, V^*, $L_\infty(\Omega)$ and $L_2(I, V)$, respectively. By $\to$ and $\rightharpoonup$, we mean strong and weak convergence. C denotes the generic positive constant.

The solution of (1.1)–(1.3) we understand in the variational sense, i.e., we look for $u\in L_2(I, V)$ with $\partial_t u\in L_2(I, V^*)$, $u(0)=u_0$ (in $L_2(\Omega)$) such that the identity

$$\langle\partial_t u, v\rangle+\left(g(|\nabla G_\sigma * u|)\nabla u, \nabla v\right)=\left(f(u-u_0), v\right) \tag{2.5}$$

holds for all $v\in V$ and for a.e. $t\in I$.

Similarly, $u_i\in V$, for $i=1,\dots,n$, is understood as the variational solution of (1.5), i.e., it satisfies the identity

$$(\delta u_i, v)+\left(g(|\nabla G_\sigma * u_{i-1}|)\nabla u_i, \nabla v\right)=\left(f(u_{i-1}-u_0), v\right) \quad \forall v\in V. \tag{2.6}$$

The existence of $u_i\in V$, $i=1,\dots,n$, from (2.6) is guaranteed by the Lax–Milgram argument.

To take the limit for $n\to\infty$ in (2.6) we shall use *a priori* estimates derived in the following lemmas.

Lemma 2.1. *The estimates*

$$\max_{1\leqslant i\leqslant n} |u_i| \leqslant C, \qquad \sum_{i=1}^{n} |\nabla u_i|^2\tau \leqslant C, \qquad \sum_{i=1}^{n} |u_i - u_{i-1}|^2 \leqslant C$$

hold uniformly for n.

Proof. First let us test (2.6) by $v = u_i\tau$ and sum it over $i = 1,\ldots,j$. We obtain

$$|u_j|^2 + \sum_{i=1}^{j} \tau \int_\Omega \alpha_{i-1}(\nabla u_i)^2 + \sum_{i=1}^{j} |u_i - u_{i-1}|^2 \leqslant C_1 + C_2 \sum_{i=1}^{j} |u_i|^2\tau, \tag{2.7}$$

where $C_1 \equiv C(u_0, f)$ and $\alpha_{i-1} \equiv g(|\nabla G_\sigma * u_{i-1}|)$. Applying Gronwall's argument in (2.7) we obtain

$$|u_i| \leqslant C, \qquad \forall n, \quad i = 1,\ldots,n.$$

Then the estimate

$$|\nabla G_\sigma * u_{i-1}|_\infty \leqslant C_\sigma |u_{i-1}| \leqslant C_\sigma \tag{2.8}$$

implies

$$g(|\nabla G_\sigma * u_{i-1}|) \geqslant \nu_\sigma > 0 \qquad \forall n, \quad i = 1,\ldots,n. \tag{2.9}$$

Then from (2.7) we obtain the assertion of Lemma 2.1. □

Together with $u^{(n)}(t)$ (see (1.6)) we consider the step function

$$\bar{u}^{(n)}(t) = u_i, \qquad \text{for } t_{i-1} < t \leqslant t_i, \quad i = 1,\ldots,n,$$

$$\bar{u}^{(n)}(0) = u_0.$$

Consequence 2.2. *The estimates*

$$\int_I \| \partial_t u^{(n)} \|_*^2 \leqslant C, \qquad \int_I \| \bar{u}^{(n)} \|^2 \leqslant C,$$

$$\int_I | u^{(n)} - \bar{u}^{(n)} |^2 \leqslant \frac{C}{n}, \qquad \max_{t\in I} | u^{(n)}(t) | \leqslant C \tag{2.10}$$

hold uniformly for n.

The estimate $(2.10)_1$ is the consequence of Lemma 2.1 and the duality argument applied in (2.6) since

$$\| \delta u_i \|_* := \sup_{v\in V, \|v\|\leqslant 1} |(\delta u_i, v)| \leqslant C_1 + C_2 \| u_i \|.$$

The estimate $(2.10)_3$ follows from the definition of $u^{(n)}$ and $\bar{u}^{(n)}$ and from the third estimate in Lemma 2.1.

Now, we shall prove the compactness of $\bar{u}^{(n)}$ in $L_2(Q_T)$, which is a consequence of the following assertion.

Lemma 2.3. *The estimate*

$$\int_0^{T-z} |\bar{u}^{(n)}(t+z) - \bar{u}^{(n)}(t)|^2 \leqslant C(z+\tau)$$

holds uniformly for $0 < z < z_0$ *and* n.

Proof. We sum up (2.6) for $i = j+1, \ldots, j+k$ considering $v = (u_{j+k} - u_j)\tau$. Then we sum it up for $j = 1, \ldots, n-k$ and obtain the estimate

$$\sum_{j=0}^{n-k} |u_{j+k} - u_j|^2 \tau \leqslant Ck\tau,$$

where Lemma 2.1 has been used. Hence for $k\tau < z < (k+1)\tau$ we deduce the desired estimate. □

Lemma 2.4. *There exists* $u \in L_2(I, V)$ *with* $\partial_t u \in L_2(I, V^*)$ *such that*

$$u^{(n)} \to u,\ \bar{u}^{(n)} \to u \quad \text{in } L_2(Q_T),$$

$$\bar{u}^{(n)} \rightharpoonup u \quad \text{in } L_2(I, V),$$

$$\partial_t u^{(n)} \rightharpoonup \partial_t u \quad \text{in } L_2(I, V^*)$$

(*in the sense of subsequences*).

Proof. The estimate $(2.10)_2$ implies

$$\int_{Q_T} \left(\bar{u}^{(n)}(t, x+y) - \bar{u}^{(n)}(t, x)\right)^2 \mathrm{d}x \leqslant C|y|, \quad \forall |y| \leqslant y_0,$$

see, e.g., [6]. Hence and from Lemma 2.3, $\{\bar{u}^{(n)}\}$ is compact in $L_2(Q_T)$ because of Kolmogorov's compactness argument (see [6]). So we can conclude $u^{(n)} \to u$ and $\bar{u}^{(n)} \to u$ in $L_2(Q_T)$ and also pointwise in Q_T. Hence and from the Consequence 2.2 we obtain the rest. □

Theorem 2.5. *There exists a unique variational solution* u *of* (1.1)–(1.3) *and* $u^{(n)} \to u$ *in* $C(I, L_2) \cap L_2(I, V)$, *where* $u^{(n)}$ *is the Rothe function defined in* (1.5)–(1.6).

Proof. We rewrite (2.6) into the form

$$\int_0^t (\partial_t u^{(n)}, v) + \int_0^t \left(g(|\nabla G_\sigma * \bar{u}_\tau^{(n)}|)\nabla\bar{u}^{(n)}, \nabla v\right) = \int_0^t \left(f(\bar{u}_\tau^{(n)} - u_0), v\right), \qquad (2.11)$$
$$\forall v \in V, \quad \forall t \in I,$$

where $\bar{u}_\tau^{(n)}(t) := \bar{u}^{(n)}(t-\tau)$. To take the limit for $n \to \infty$ in (2.11) we apply Lemma 2.4 and the fact that $\bar{u}_\tau^{(n)} \to u$ in $L_2(Q_T)$ and hence $g(|\nabla G_\sigma * \bar{u}_\tau^{(n)}|) \to g(|\nabla G_\sigma * u|)$ a.e. in Q_T. Then we have

$$\int_0^t \langle \partial_t u, v\rangle + \int_0^t (g(|\nabla G_\sigma * u|)\nabla u, \nabla v) = \int_0^t (f(u-u_0), v)$$
$$\forall v \in V, \quad \forall t \in I,$$

where u is the same as in Lemma 2.4. So u is the variational solution of (1.1)–(1.3). Its uniqueness can be obtained by the same arguments as used in [3]. As a consequence we have that the original sequences $\{u^{(n)}\}$ and $\{\bar{u}^{(n)}\}$ converge to u.

Now we shall prove the $C(I, L_2) \cap L_2(I, V)$ convergence of $u^{(n)}$ to u. For this purpose we verify that

$$\int_0^t \left(\partial_t u^{(n)}, \bar{u}^{(n)} - u\right) \geqslant \mathrm{o}(1) \tag{2.12}$$

(the Landau symbol $\bar{\mathrm{o}}(1)$ represents a term c_n satisfying $c_n \to 0$ for $n \to \infty$), which is the consequence of the following facts:

$$\sum_{i=1}^{j} (u_i - u_{i-1}, u_i) = \tfrac{1}{2}|u_j|^2 - \tfrac{1}{2}|u_0|^2 + \tfrac{1}{2}\sum_{i=1}^{j} |u_i - u_{i-1}|^2,$$
$$\partial_t u^{(n)} \rightharpoonup \partial_t u \quad \text{in } L_2(I, V^*),$$

the integration by parts formula

$$\int_0^{t_j} \langle \partial_t u, u\rangle = \tfrac{1}{2}|u(t_j)|^2 - \tfrac{1}{2}|u_0|^2 \tag{2.13}$$

and $u^{(n)} \to u$ in $L_2(Q_T)$.

So, let us test (2.6) by $v = \bar{u}^{(n)} - u$ and integrate it over $(0, t)$. Then using (2.12) and

$$\int_0^t \left(g(|\nabla G_\sigma * \bar{u}_\tau^{(n)}|)\nabla u, \nabla(\bar{u}^{(n)} - u)\right) = \mathrm{o}(1)$$

(since $g(|\nabla G_\sigma * \bar{u}_\tau^{(n)}|)\,\nabla u \to g(|\nabla G_\sigma * u|)\nabla u$ in $L_2(Q_T)$) we conclude

$$\nu_\sigma \int_0^t |\nabla(\bar{u}^{(n)} - u^{(n)})|^2 \leqslant \int_0^t \left(g(|\nabla G_\sigma * \bar{u}_\tau^{(n)}|)\nabla(\bar{u}^{(n)} - w), \nabla(\bar{u}^{(n)} - u)\right) \leqslant \mathrm{o}(1),$$

which implies $\bar{u}^{(n)} \to u$ in $L_2(I, V)$.

To prove $u^{(n)} \to u$ in $L_2(I, V)$ we use the auxiliary function

$$\bar{u}_{(n)}(t) := \bar{u}_{i-1} + \frac{(t - t_{i-1})}{\tau}\left(\bar{u}_i - \bar{u}_{i-1}\right), \qquad \text{for } t_{i-1} \leqslant t \leqslant t_i, \quad i = 1, \ldots, n$$

where

$$\bar{u}_i := \frac{1}{\tau}\int_{t_{i-1}}^{t_i} u(t)\,\mathrm{d}t.$$

From the Lebesgue theorem (continuity in mean), it follows that $\bar{u}_{(n)} \to u$ in $L_2(I, V)$. Then we estimate

$$\int_I \| u^{(n)} - \bar{u}_{(n)} \|^2 \leqslant C \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \| u_i - \bar{u}_i \|^2$$
$$\leqslant C \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\| u_i - u(s) \|^2 + \| \bar{u}_i - u(s) \|^2 \right)$$
$$\leqslant C\left(\| \bar{u}^{(n)} - u \|^2_{L_2(I,V)} + \| \bar{u}_{(n)} - u \|^2_{L_2(I,V)} \right) \to 0,$$

from which the required assertion follows.

Now we integrate (2.5) over $(0, t)$ and subtract it from (2.11) where $v = u^{(n)} - w$. Then using the integration by parts formula (2.13) we obtain

$$| u(t) - u^{(n)}(t) |^2 \leqslant \int_0^t \left(g\left(| \nabla G_\sigma * \bar{u}_\tau^{(n)} | \right) \nabla (u - \bar{u}^{(n)}),\ \nabla (u - u^{(n)}) \right)$$
$$+ \int_0^t \left(| g\left(| \nabla G_\sigma * \bar{u}_\tau^{(n)} | \right) - g\left(| G_\sigma * u | \right) |_\infty | \nabla (u - \bar{u}^{(n)}) | \, | \nabla (u - u^{(n)}) | \to 0 \right.$$

since $\bar{u}^{(n)} \to u$ in $L_2(I, V)$. Thus the proof is complete. □

3. Full discretization scheme

The convergence results obtained in the previous section can be extended to the full discretization scheme, in which (2.6) is projected on a finite-dimensional subspace $V_\lambda \subset V$. We assume $V_\lambda \to V$ for $\lambda \to 0$ in a canonical sense, i.e.

$$\forall v \in V \ \exists v_\lambda \in V_\lambda \text{ such that } v_\lambda \to v \text{ for } \lambda \to 0 \text{ in } V. \tag{3.1}$$

Instead of $u_i \in V$ we look for $u_i^\lambda \in V_\lambda$, $i = 1, \ldots, n$, such that

$$\begin{aligned} &\left(\delta u_i^\lambda, v \right) + \left(g\left(| \nabla G_\sigma * u_{i-1}^\lambda | \right) \nabla u_i^\lambda, \nabla v \right) = \left(f\left(u_{i-1}^\lambda - u_0 \right), v \right) \quad \forall v \in V_\lambda, \\ &\partial_\nu u_i^\lambda = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.2}$$

By the same arguments as in Section 2 we obtain the existence of $u_i^\lambda \in V_\lambda$, for $i = 1, \ldots, n$, and *a priori* estimates (see Lemma 2.1)

$$\max_{1 \leqslant i \leqslant n} | u_i^\lambda | \leqslant C, \qquad \sum_{i=1}^{n} | \nabla u_i^\lambda |^2 \tau \leqslant C, \qquad \sum_{i=1}^{n} | u_i^\lambda - u_{i-1}^\lambda |^2 \leqslant C, \tag{3.3}$$

which hold uniformly for n and λ.

Let $u_0^\lambda \in V_\lambda$ be such that $u_0^\lambda \to u_0$ in $L_2(\Omega)$ for $\lambda \to 0$. By means of u_i^λ, $i = 0, \ldots, n$, we define the functions $u^{(\alpha)}(t)$ and $\bar{u}^{(\alpha)}(t)$, $\alpha = (\tau, \lambda)$, in the same way as the functions $u^{(n)}$ and $\bar{u}^{(n)}$ have been constructed by means of u_i (see (1.6)). Then (3.3) can be rewritten in the form

$$\int_I \| \partial_t u^{(\alpha)} \|^2_{V_\lambda^*} \leqslant C, \qquad \int_I \| \bar{u}^{(\alpha)} \|^2 \leqslant C, \qquad \int_I | u^{(\alpha)} - \bar{u}^{(\alpha)} |^2 \leqslant C\tau, \tag{3.4}$$

which holds uniformly for α. The estimate $(3.4)_1$ has been obtained by the duality argument in (3.2) taking into account $v \in V_\lambda$.

Along the same lines as in Lemma 2.3 we obtain that the estimate

$$\int_0^{T-z} |\bar{u}^{(\alpha)}(t+z) - \bar{u}^{(\alpha)}(t)|^2 \leqslant C(z+\tau) \tag{3.5}$$

holds uniformly for $0 < z < z_0$ and α.

Now we can formulate

Theorem 3.1. *Let $u_0^\lambda \in V_\lambda$ be such that $u_0^\lambda \to u_0$ in $L_2(\Omega)$ for $\lambda \to 0$. If* (2.2)–(2.4) *and* (3.1) *are satisfied, then $u^{(\alpha)} \to u$ in $C(I, L_2) \cap L_2(I, V)$ for $\alpha \to 0$, where u is the variational solution of* (1.1)–(1.3) *and $u^{(\alpha)}$ is obtained from* (3.2).

Proof. The a priori estimates (3.3)–(3.5) imply for $\alpha \to 0$ that $u^{(\alpha)} \to w$, $\bar{u}^{(\alpha)} \to w$ and $\bar{u}_\tau^{(\alpha)} \to w$ in $L_2(Q_T)$ and $\bar{u}^{(\alpha)} \rightharpoonup w$ in $L_2(I, V)$. We shall prove that $w \equiv u$ is the variational solution of (1.1)–(1.3). For this purpose we extend the functional $\partial_t u^{(\alpha)} \in L_2(I, V_\lambda^*)$ to $F^{(\alpha)} \in L_2(I, V^*)$ by the prescription

$$\int_I \langle F^{(\alpha)}, v \rangle := \int_I \langle \partial_t u^{(\alpha)}, P_\lambda v \rangle = \int_I \int_\Omega \partial_t u^{(\alpha)} P_\lambda v,$$

where $P_\lambda : V \to V_\lambda$ is the orthogonal projector. Thus $\| F^{(\alpha)} \|_{L_2(I,V^*)} \leqslant C$ and $F^{(\alpha)} \rightharpoonup F \in L_2(I, V^*)$. Since $u^{(\alpha)} \to w$ in $L_2(Q_T)$ we obtain $F \equiv \partial_t w$ (see, e.g., [4]). Now for fixed $v \in V$ we choose $v_\lambda \in V_\lambda$ with $v_\lambda \to v$ in V for $\lambda \to 0$. We test (3.2) with $v = v_\lambda$, integrate it over $(0, t)$ and take the limit for $\alpha \to 0$. Then the first term gives

$$\int_0^t \int_\Omega \partial_t u^{(\alpha)} v_\lambda = \int_0^t \langle \partial_t u^{(\alpha)}, v_\lambda \rangle = \int_0^t \langle F^{(\alpha)}, P_\lambda v \rangle \to \int_0^t \langle F, v \rangle \equiv \int_0^t \langle \partial_t w, v \rangle.$$

By the same arguments as in Section 2 we obtain that w is a variational solution of (1.1)–(1.3). The uniqueness argument guarantees that $u \equiv w$ and the original sequences $u^{(\alpha)}$ and $\bar{u}^{(\alpha)}$ are convergent. To prove $u^{(\alpha)} \to u$ in $C(I, L_2)$ and $\bar{u}^{(\alpha)} \to u$ in $L_2(I, V)$ we proceed in the same way as in the proof of Theorem 2.5 using the test function $v = \bar{u}^{(\alpha)} - w^{(\alpha)}$, where $w^{(\alpha)} \in L_2(I, V_\lambda)$ with $w^{(\alpha)} \to u$ in $L_2(I, V)$. Then using the inequality

$$\int_0^t \langle \partial_t u^{(\alpha)}, \bar{u}^{(\alpha)} - w^{(\alpha)} \rangle \geqslant o(1)$$

we obtain the rest of our assertion along the same lines as in the proof of Theorem 2.5. □

4. Numerical experiments

We shall describe several numerical experiments computed by the approximation scheme (1.5)–(1.6) in order to process the initial picture by nonlinear diffusion (1.1)–(1.3). In our practical implementation the picture consists of small square pixels on which u_{i-1} is constant. Then the convolution $G_\sigma * u_{i-1}$ is reduced to a weighted mean value on neighbour pixels. We have used σ so small that this weighted mean value is realized only on some neighbour squares (0, 1, ... ,4, etc.) since the weights generated by the Gauss function are machine zeroes on pixels with bigger distance. In the presented experiments, we use for image representation 70×70 points, and the function $g(s) = 1/(1+s^2)$.

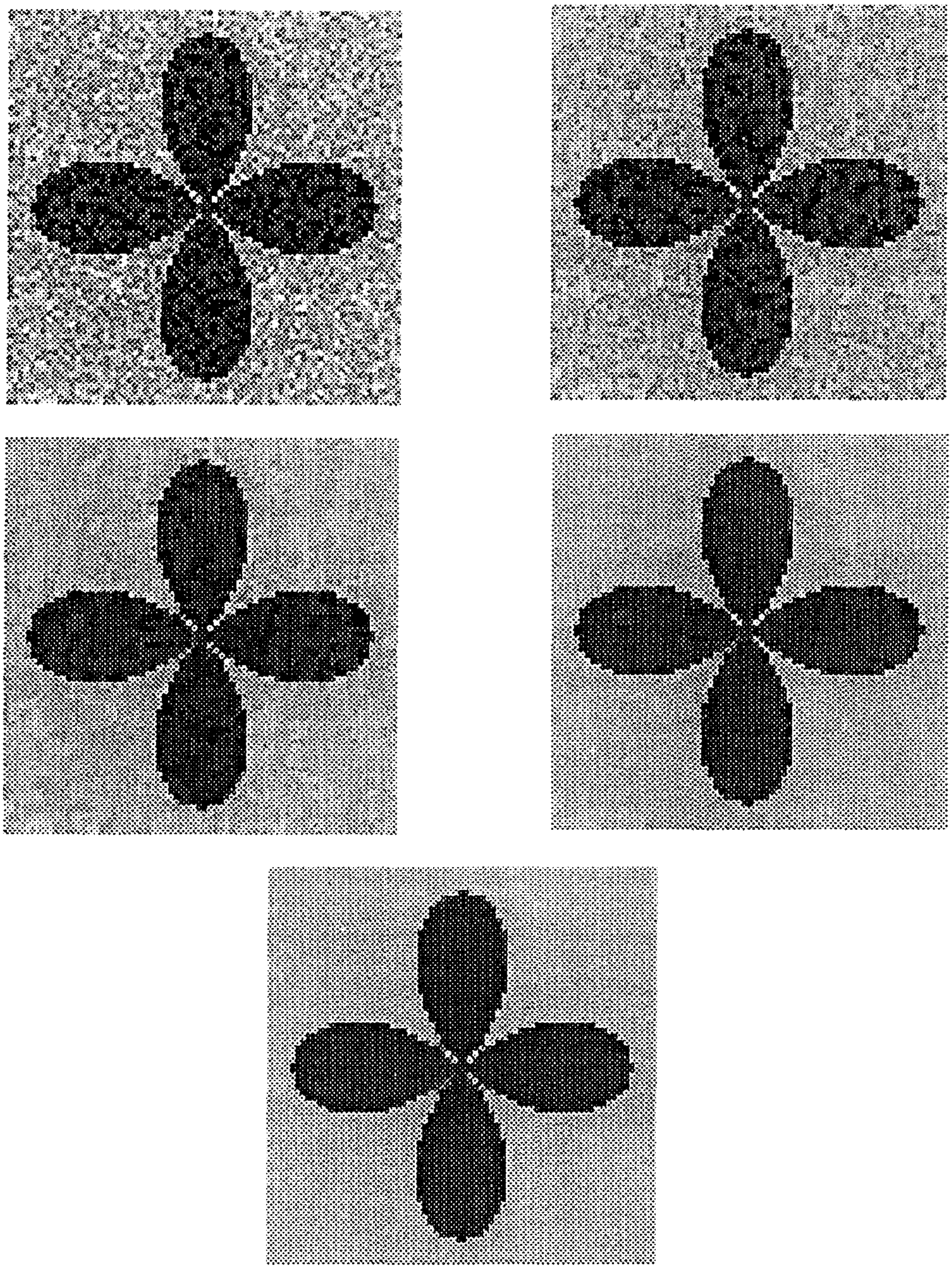

Fig. 1.

In Fig. 1 a randomly destroyed initial shape is restored by selective diffusion (1.1)–(1.3) (here and in next two experiments $\sigma = 10^{-8}$). There are plotted time steps $t = 0., 0.01, 0.02, 0.03, 0.04$ of the evolution, computed with numerical time step $\tau = 0.005$ (which is also the same as in the next two experiments).

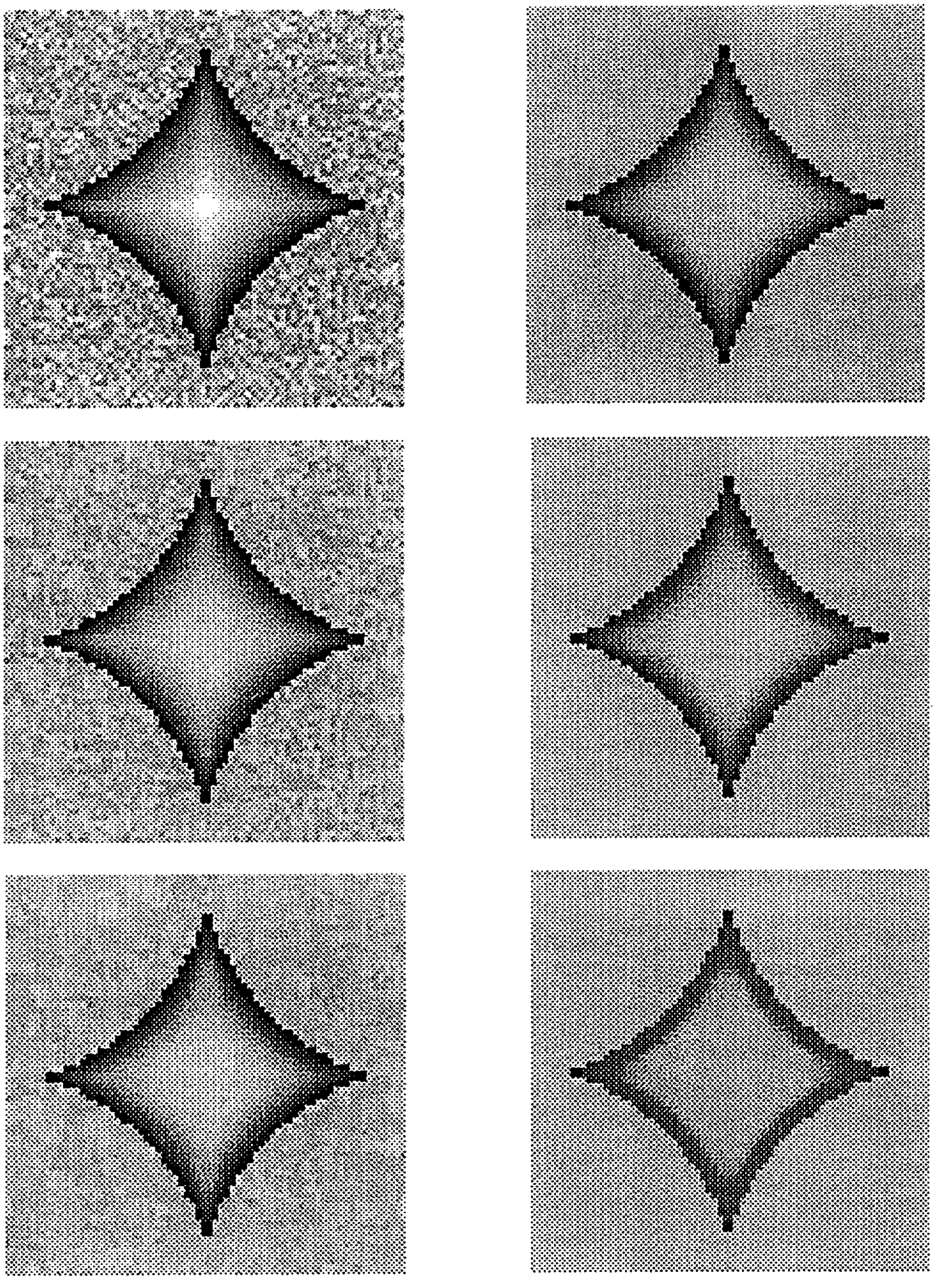

Fig. 2.

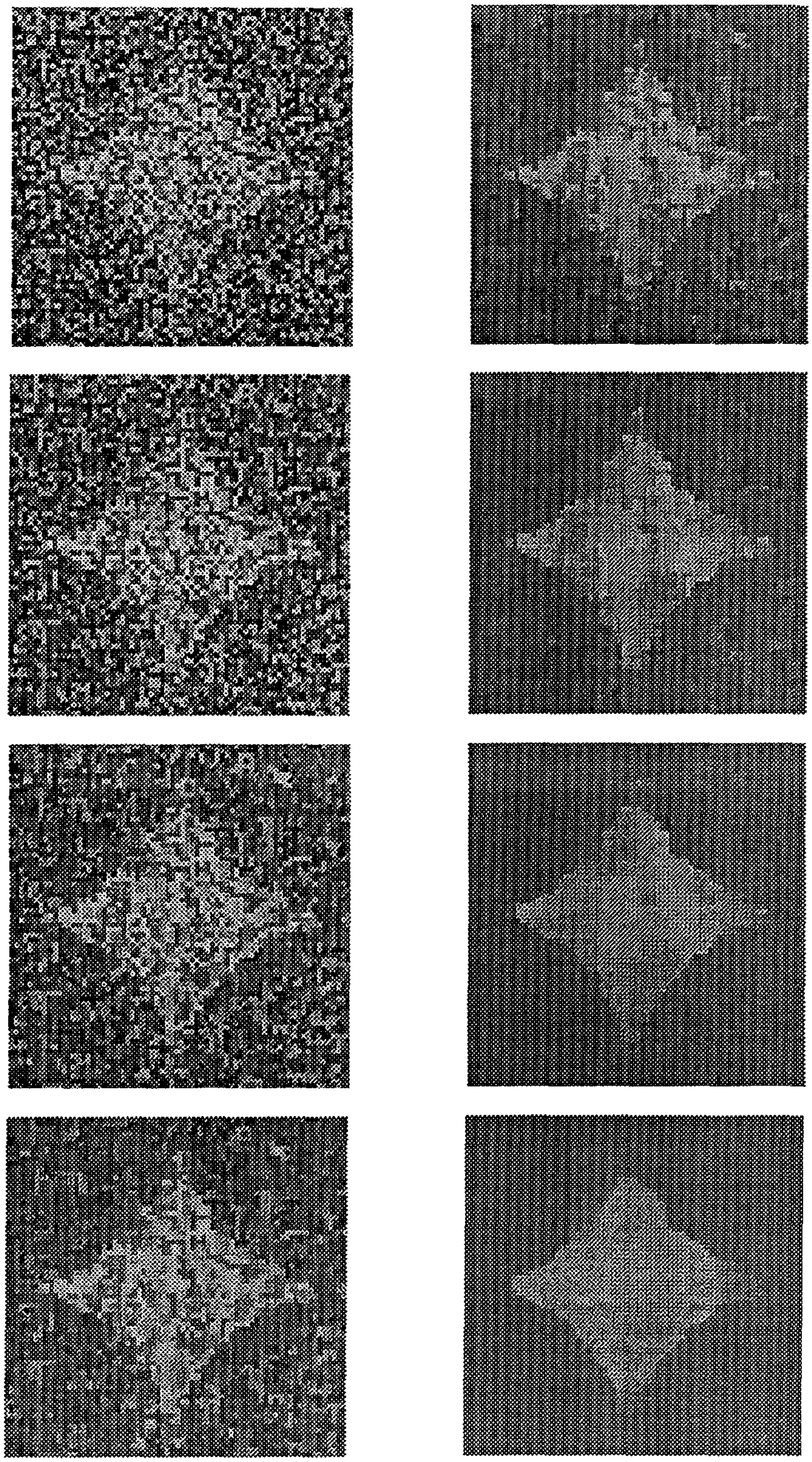

Fig. 3.

A similar experiment is presented in Fig. 2. Here we plot the time moments $t =$ 0., 0.01, 0.02, 0.03, 0.04, 0.10.

In Fig. 3 a it is tried to restore strongly destroyed image. We plot the solution in time moments $t = 0., 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.08$.

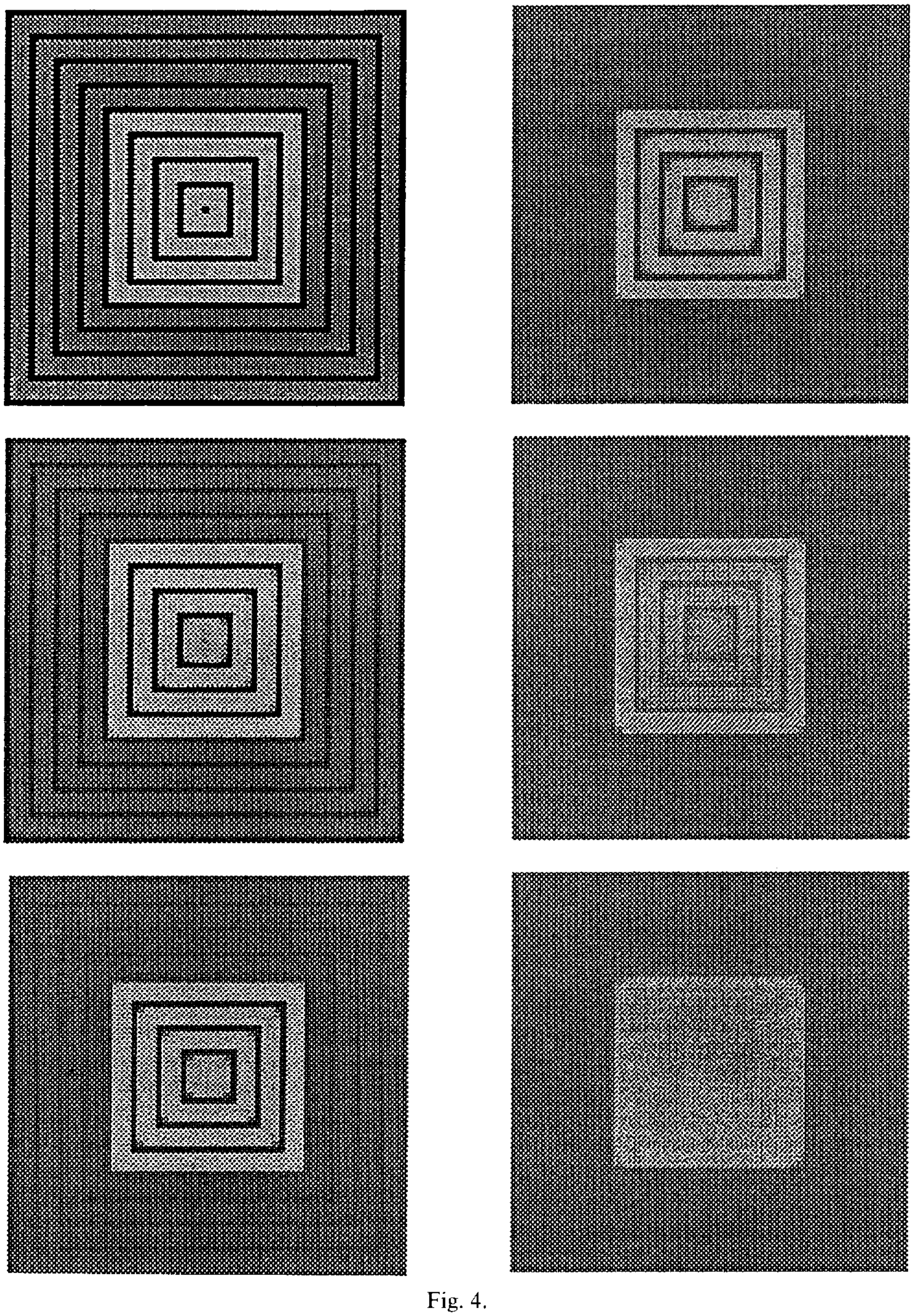

Fig. 4.

In Fig. 4 the spreading of the structural noise by selective diffusion with $\sigma = 3 \cdot 10^{-5}$, $g(s) = 1/(1+s^2)$, can be recognized. The plotted states are in $t = 0.$, 0.04, 0.06, 0.08, 0.10, 0.12 ($\tau = 0.01$).

References

[1] L. Alvarez, F. Guichard, P.L. Lions and J.M. Morel, Axioms and fundamental equations of image processing, *Arch. Rat. Mech. Anal.* 123 (1993) 200–257.
[2] L. Alvarez, P.L. Lions and J.M. Morel, Image selective smoothing and edge detection by nonlinear diffusion II, *SIAM J. Numer. Anal.* 29 (1992) 845–866.
[3] F. Catté, P.L. Lions, J.M. Morel and T. Coll, Image selective smoothing and edge detection by nonlinear diffusion, *SIAM J. Numer. Anal.* 29 (1992) 182–193.
[4] H. Gajewski, K. Gröger and K. Zacharias, *Nichtlineare Operatorgleichnungen und Operatordifferentialgleichungen* (Academia-Verlag, Berlin, 1974).
[5] A. Kufner, O. John and S. Fučík, *Function Spaces* (Academia, Prague, 1977).
[6] J. Nečas, *Les Methodes Directes en Théorie des Équations Elliptiques* (Academia, Prague, 1967).
[7] K.N. Nordström, Biased anisotropic diffusion—a unified approach to edge detection, Preprint, Department of Electrical Engeneering and Computer Sciences, University of California, Berkeley, CA (1989).
[8] P. Perona and J. Malik, Scale space and edge detection using anisotropic diffusion, in: *Proceedings IEEE Computer Society Workshop on Computer Vision* (1987).