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WHAT WAS THE RIVER ISTER
IN THE TIME OF STRABO?
A MATHEMATICAL APPROACH

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ABSTRACT. We introduce a novel method for map registration and apply it to transformation of the river Ister from *Strabo's map of the World* to the current map in the World Geodetic System. This transformation leads to the surprising but convincing result that Strabo's river Ister best coincides with the nowadays Tauernbach-Isel-Drava-Danube course and not with the Danube river what is commonly assumed. Such a result is supported by carefully designed mathematical measurements and it resolves all related controversies otherwise appearing in understanding and translation of Strabo's original text. Based on this result, we also show that *Strabo's Suevi in the Hercynian Forest* corresponds to the Slavic people in the Carpathian-Alpine basin and thus that the compact Slavic settlement was there already at the beginning of the first millennium AD.

1. Introduction

In this paper, we present a novel mathematical model and numerical method for the transformation of geographic maps to each other. To be more precise, we are interested in finding the transformation of a historical map to a current one in the World Geodetic System (WGS) [42]. Such a problem is also called map registration. Our mathematical model and numerical method for the map registration is based on two main principles and steps.

First, we design and compute locally optimal affine transformations of one map to the other. In this step, every locally optimal affine transformation is found by means of the least square method using a set of clearly identified corresponding points.

Then, in the second step, the locally optimal affine transformations are smoothly interpolated/extrapolated to all other points of the map by solving the Laplace equation with suitable boundary conditions. The solution of the Laplace equation is obtained numerically by using the finite difference method on a background grid discretizing the selected rectangular region of interest on the map. After obtaining the final transformation, which we call *Locally Affine Globally Laplace* (LAGL) map transformation, we can transform any point of the historical map to the current map and see which places on the current map correspond to geographic objects on the historical map.
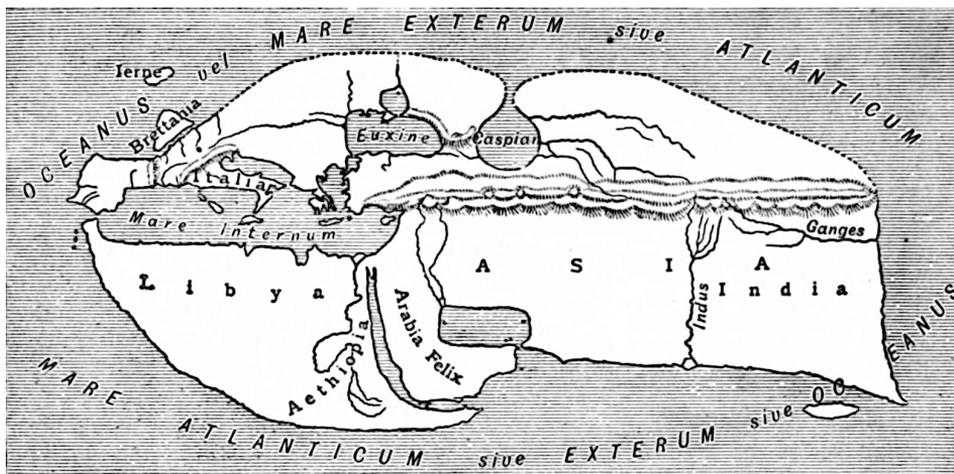

FIGURE 1.1. Strabo's map of the World.

The motivation to study this problem mathematically and numerically comes to us by reading Strabo's *Geographica* (Στράβωνος Γεωγραφικά) and studying the related *Strabo's map of the World* published in the Encyclopaedia Biblica [10], in the section "Geography" on page 1691, see Figure 1.1. *Strabo's map of the World* is authored by Karl Müller, a famous 19th-century historian, classic philologist, geographer, and cartographer, who translated Strabo's *Geographica* from Greek to Latin in 1853 [19]. This book is one of our reading sources because it contains the Greek text without "purposeful" changes found sometimes in other sources and translations. The further very useful Greek text of Strabo's *Geographica*, allowing direct translation of Greek words in English is available

in Perseus Digital Library [34] of Tufts University. It contains the *Geographica* edition by August Meineke from 1877 [20] but one has to be careful because at some points it deviates from [19] and other sources, e.g., by exchanging river names or their transcript with respect to the original. The most recent and very useful source for reading is the English translation of Strabo's work by Duane W. Roller [30] where he aims to respect Strabo's original geographic names and he does not translate them to commonly used nowadays terms. Concerning the translation and understanding of the Strabo's work, it is worth to cite Roller's book, Section 5: "there is still the problem of many rare or unique words, extensive paraphrases of earlier authors who are themselves obscure, ambiguities of style and sheer length of the work". There exist some further useful translations which can be found on the internet, e.g. [35], suitable for an introductory reading.

Strabo was a Greek geographer who lived from around 63 BC to around AD 24 [30]. He lived in Asia Minor, Rome and Alexandria and travelled a lot during his lifetime to collect the information for his work not only by reading preceding sources such as Eratosthenes and others, mainly in the famous Alexandria library. Strabo's *Geographica* was first published in 7 BC, collecting all the knowledge from the previous years, and he continued the work until approximately AD 23, during the reigns of the emperors Augustus and Tiberius. Strabo's *Geographica* is considered to be one of the rare ancient scientific works in the human history remained to modern times, it is not a historical narrative, but it gives a huge amount of useful quantified information about the known world at the beginning of the first millennium AD. It is very important to note that we must not apply any later knowledge of the Romans about Europe and the World when reading *Geographica*.

As it is announced in the title of the paper, we investigate how the river Ister (Ἴστρος) is transformed from the historical *Strabo's map of the World* to the current map of the world. We discovered an astonishing fact that the **Strabo's river Ister**, or better say the river Ister in the times of Strabo, does not correspond to the river Danube on its entire course, but it **perfectly fits with** the nowadays **Tauernbach-Isel-Drava-Danube** course (or if simplified we can just say to the Drava-Danube course). This result is surprising but convincing, supported by carefully designed quantitative mathematical measurements. First, by computing a distance of sources of the current Danube, Drava, Isel and Tauernbach rivers and the source of the transformed Strabo's Ister. Further, by computing the Hausdorff distance of curves representing the respective river courses. And finally, by computing the common length of the compared river courses in prescribed narrow bands. From all these comparisons and from Strabo's writing itself it is clear that the current Tauernbach-Isel-Drava-Danube course gives the best correspondence with the Ister in the Strabo's times. Moreover, this
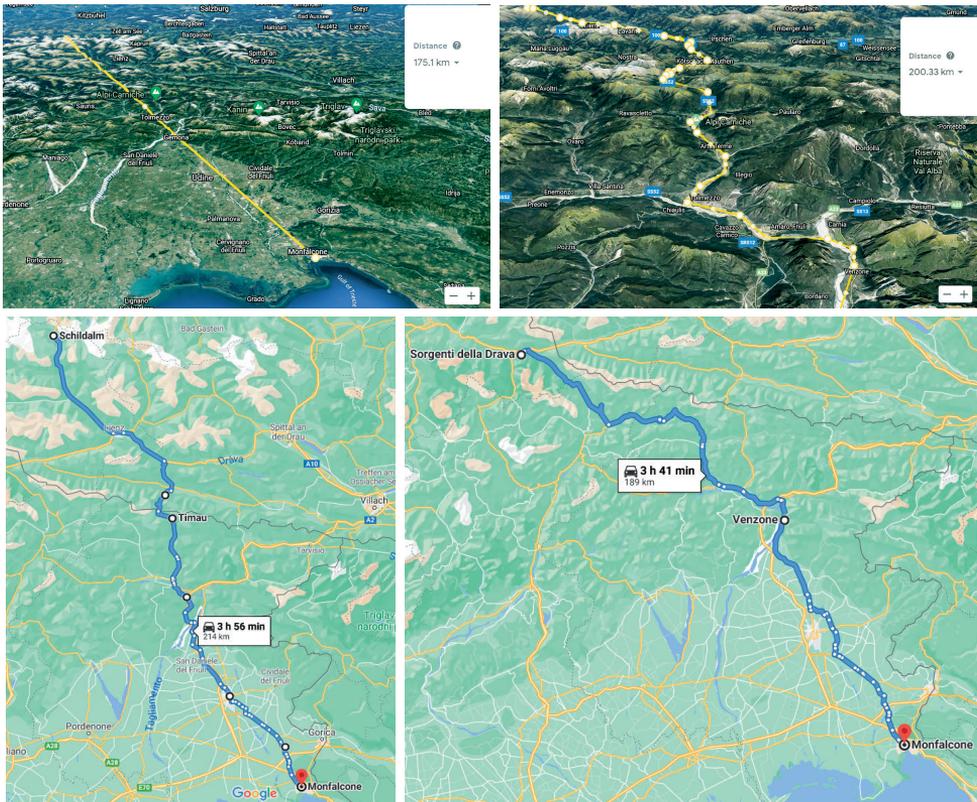
FIGURE 1.2. Distances to the recess of Adriatic (Monfalcone). Left up: geodesic line distance from the source of Tauernbach in the High Tauern (175 km). Right up: geodesic distance from the source of the Tauernbach through the passable valleys (200 km). Left down: distance from the source of the Tauernbach by local roads (214 km). Right down: distance from the source of the Drava river by local roads (189 km). Upper images were created by the Google Earth application while bottom images by the Google Maps application.

result avoids several, if not all, contradictions which otherwise occur when reading carefully *Geographica* and its later translations which used to consider the Ister as the Danube river in the whole its course, i.e., with the sources in the Schwarzwald, Germany. In the sequel, we mention briefly just a few, the most important contradictions, and their solution given by our mathematical result.

First, hardly explainable contradictions are given by the location of the Ister source, its distance from the recess of Adriatic and by the direction of the Ister course itself, as described by Strabo in the Book 7, Part 1, Section 1 of *Geographica*, which we denote by (7–1–1) and other sections are denoted in the same manner. In (7–1–1) Strabo says:

> "Ἴστρος ... ῥέων πρὸς νότον κατ᾽ ἀρχάς, εἶτ᾽ ἐπιστρέφων εὐθὺς ἀπὸ τῆς δύσεως ἐπὶ τὴν ἀνατολὴν καὶ τὸν Πόντον. ἄρχεται μὲν οὖν ἀπὸ τῶν Γερμανικῶν ἄκρων τῶν ἑσπερίων, πλησίον δὲ καὶ τοῦ μυχοῦ τοῦ Ἀδριατικοῦ, διέχων αὐτοῦ περὶ χιλίους σταδίους".

First of all, at the end of the second sentence, Strabo clearly says that the distance of the Ister source from the recess of Adriatic is about *1000 stadia* (χιλίους σταδίους). The stadium corresponds to 177.7–197.3 m, see [30, page 33]. Our mathematical result, presented in the next sections of the paper, shows that the source of the Ister corresponds either to the source of Drava in Sorgenti della Drava in the Val di Pusteria or to the Isel or Tauernbach sources in the High Tauern, see Figure 1.4 (this Figure and some further were created by the help of software [41]). We note that the source of the Isel river was in the past considered approximately in the place of the source of nowadays Tauernbach creek and that such an interconnected stream was called the *Isola flu* on historical maps [3, 21], see Figure 1.3. But let us consider the nowadays situation and measure distances from the recess of Adriatic, placed into Monfalcone close to the ancient Roman city Aquileia, to those three river sources. First, let us measure the distance to the Tauernbach source. We placed the Tauernbach source to the highest possible point below the Großvenediger (called the Windisch Taurn on Figure 1.3) on its north-east side. The direct geodesic distance (the shortest path on the Earth surface) to the Monfalcone is about 175 km and the geodesic distance through the passable valleys is about 200 km, see Figure 1.2 upper row. We also measured the distance by using the local roads to the nearest place to the source, in Schildalm, and we got the distance equal to 214 km, see Figure 1.2 left bottom. With a high probability, such travel may fit very well with the way to that places in the Strabo's time through the Alpine valleys. And we see that all these distances are in perfect agreement with the Strabo's information about approximately 1000 stadia from the recess of Adriatic! When we considered the distances by the local roads to the other two sources, of the Isel in the Hinterbichl at the end of the valley just south of the Großvenediger, and of the Drava in the Val di Pusteria we got 216 km and 189 km, see Figure 1.2, respectively. They are again very good estimates of 1000 stadia. This cannot happen in any case when considering the source of Ister in the source of Danube in Schwarzwald, Germany, with a distance to the recess of Adriatic about 640 km, approximately 3500 stadia, highly exceeding the Strabo's *Geographica* information.

Further, in the first part of the second sentence of the cited text, Strabo says that the Ister makes its beginning from the western highest summits (ἄκρων) of the "Germani" people. Indeed, Tauernbach has its spring exactly between the Großvenediger and Großglockner, two highest peaks of the High Tauern, see Figure 1.4, and of all the north-eastern Alps, thus it again perfectly fits with the Strabo's description. Here we note that Strabo also explicitly writes that the "germani" means "genuine" (γνησίους) in his time Roman language. This is clearly stated in the translation of the last sentence of Section (7–1–2):

"γνήσιοι γὰρ οἱ Γερμανοὶ κατὰ τὴν Ῥωμαίων διάλεκτον"

by Roller [30], and also from further context of *Geographica* it is clear that, in general, Strabo uses the term "Germani" to denote all genuine people east of the Rhine (Ῥῆνος) and north and east of the Alps (including the north-eastern Alps themselves). The term "Germani" represents a much wider notion than the 19th century and the nowadays concept of Germans and their language.
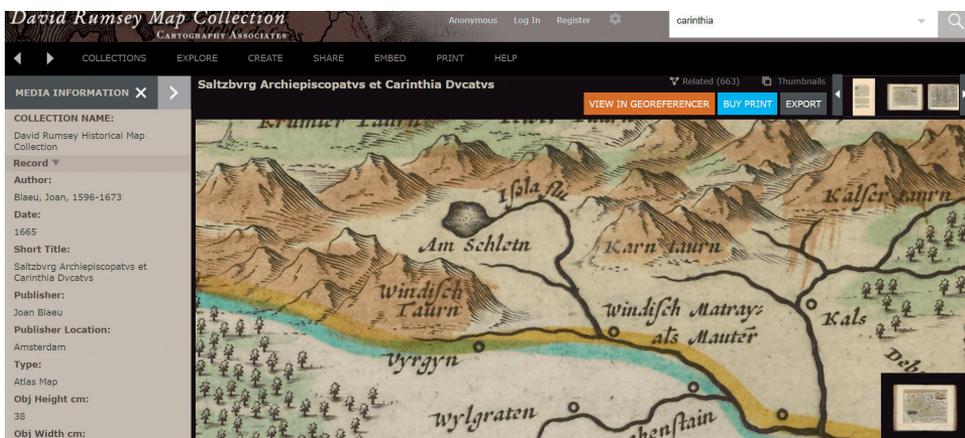
FIGURE 1.3. Detail of the map by Joan Blaeu from 1665 [3], where we see the High Tauern and the Isola flu corresponding to the Tauernbach--Isel stream. The Großvenediger corresponds to the Windisch Taurn and the Großglockner to the Kalser Taurn.

All in all, from the above facts it is clear that the location of the Ister source in a close neighbourhood of the High Tauern is the only possibility fulfilling consistently both Strabo's requirements - to be near the highest summits of the Alps east of the Rhine and to be about 1000 stadia from the recess of Adriatic. Moreover, in the first sentence of the cited Greek text above, Strabo writes that the Ister first flows to the south and soon it changes direction from the west to the east up to the Black Sea. This also perfectly corresponds to the Tauernbach--Isel-Drava-Danube course. It flows about 40 km to the south and then in Lienz,
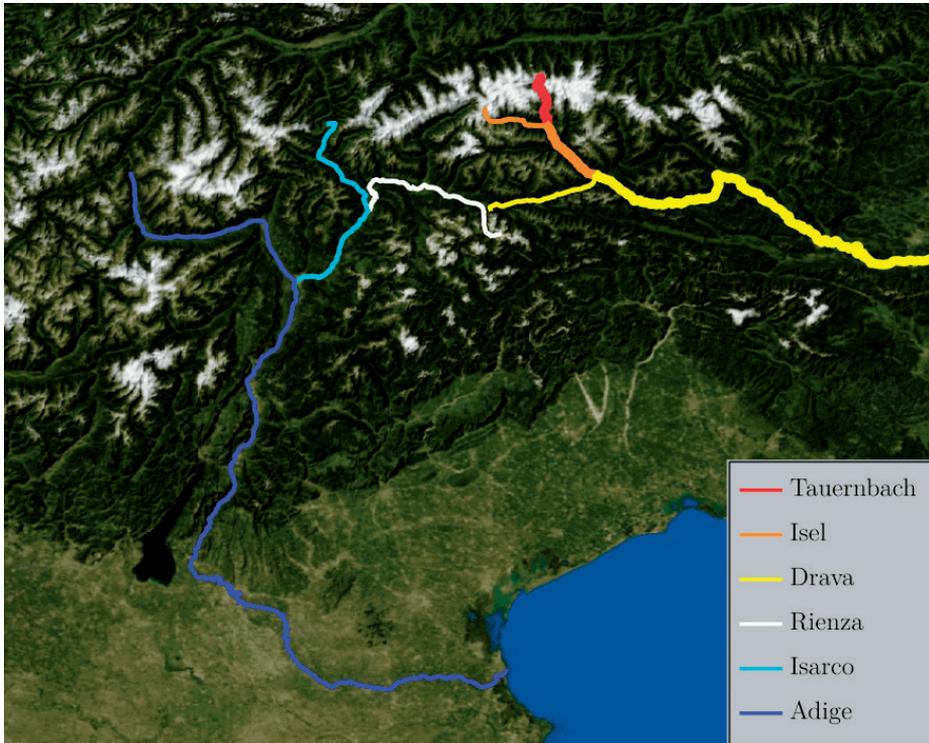
FIGURE 1.4. Detail of river courses of Rienza, Isarco and Adige, (white--cyan-blue) and the river Ister by our result, corresponding to the Tauernbach-Isel-Drava-Danube course (red-orange-yellow), plotted thicker.

after the confluence with the Drava, the course direction is changing from the west to the east. None of these facts can be derived for the Danube river from its source in the Schwarzwald, Germany.

The second, very important contradiction between placing the Ister source to the Schwarzwald, Germany and Strabo's *Geographica* occurs in Section (4–6–9). The whole Book 4, Part 6 (4–6) is devoted to a detailed description of the region of Alps from the Savona (Σαβάτα) in Liguria, Italy up to the Nanos plateau (Ὄκρᾳ) in Inner Carniola, Slovenia. Strabo's description of the Alps (Ἄλπεις, Ἄλπεια), many times called also Albia (Ἄλβια) in (4–6), follows first the direction from the south to the north, i.e., from Savona up to the Alpine part of the river Rhine. Then the description turns to the east, see (4–6–8) and (4–6–9), and going very consistently through the countries (even nowadays federal states) and mountainous regions of Ellvettians (Switzerland‐Swiss‐Helvetica), Boians

(Bavaria-Bayern), Rhaetians (Tyrol and South Tyrol), Noricians (Salzburg and Upper Austria), Tauriskians (Styria-Steiermark) up to Karnians (Carinthia- -Kärnten and Carniola-Krain), not far from the recess of Adriatic, above the territory of Karnians, Strabo arrives at the mountainous places where the source of Ister is located. In this part of Section (4–6–9), Strabo describes the river Isaras (Ἰσάρας) which after joining with the river Atagis (Ἄταγις) empties into the Adriatic. Clearly, Atagis corresponds to river Adige and Isaras corresponds to nowadays river Isarco - or most probably - to the course of Isarco continuing in Bressanone upstream by its (larger) tributary Rienza stemming from the Dolomites and flowing through Val di Pusteria, see Figure 1.4. And in these places also Ister (Ἴστρος) takes its beginning, Strabo says explicitly. It is very clear that all the sources, of Tauernbach, Isel or Drava, fulfils this geographic requirement. On the other hand, placing the source of Ister in the Schwarzwald, Germany cannot solve in any way such geographic situation of the Ister flowing from its source to the Black Sea and the neighbouring rivers flowing to the Adriatic. We see that our mathematical result brings the straightforward solution to this tedious and long-lasting controversy which yielded many troubles in translations of *Geographica*, even by exchanging the names of the rivers in Section (4–6–9) compared to original, see [19], where such possible "purposeful" changes were only indicated in brackets in the Latin translation and [20], where such changes of river names were even performed in the Greek text.

Just as a curiosity we mention that our result simply solves also the otherwise unexplainable paradox in the *voyage of Argonauts*, where, by Apollonius of Rhodes (3rd century BC) in his epic poem *Argonautica*, "the Argonauts had been obliged to abandon their regular course from Colchis homeward, and had gone from Euxine Sea (Black Sea) up the Ister and then passing down the other branch of that river, they had entered into Adriatic" [13]. By our result it is allowed, Argonauts could follow the Danube-Drava upstream up to the Val di Pusteria and continue to the Adriatic by the stream of Rienza-Isarco-Adige. The stream of Rienza is only 3 km away from the source of the Drava in Val di Pusteria where the watershed between the Black Sea and the Adriatic is located, see Figure 1.4. Of course, we do not want to claim that Argonauts made such a voyage :-) but to explain why "such story was accepted even by so able geographer as Eratosthenes who seems to have been a firm believer in the reality of the Argonautic voyage" [13]. There is no controversy and it may indicate that Eratosthenes was aware of the watershed in this Alpine region. Moreover, if we adopt a hypothesis that the notion of Ister was evolved in time as Greeks and Romans explored Europe between the Adriatic and the Black Sea from the south, we see also further "earlier" possibility of the Argonauts' voyage, following upstream the Danube-Sava course up to Zelenci and continuing near Tarvisio to the Adriatic by the Fella-Tagliamento stream.

After the above explanations of solving main geographical controversies appearing when reading *Geographica* and its later translations, let us begin our mathematical story.

It will be explained with all mathematical and computational details in the next Sections 2 and 3. Moreover, since we have to shift the sources of Ister from Schwarzwald to the High Tauern and the upper course of the Ister from the upper Danube to the Drava course, many historical facts from the beginning of the first millennium should be "shifted" as well. That opens many questions which will be discussed in Section 4. The paper will be finished by our conclusions in Section 5.

2. Locally affine globally laplace (LAGL) map transformation

To register the map M_1 to map M_2, we use an affine transformation. In general, an affine transformation is given by the formula

$$\mathbf{y} = \boldsymbol{A}\,\mathbf{x} + \mathbf{b}, \tag{2.1}$$

where $\mathbf{x} = (x_1, x_2)$ is a point on the map M_1, and $\mathbf{y} = (y_1, y_2)$ is a point on the map M_2 and

$$\boldsymbol{A} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \tag{2.2}$$

is 2×2 matrix and

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{2.3}$$

is a translation vector. For simplicity, we can write the affine transformation as follows

$$\begin{aligned} y_1 &= a_1 x_1 + a_2 x_2 + b_1, \\ y_2 &= a_3 x_1 + a_4 x_2 + b_2. \end{aligned} \tag{2.4}$$

Our goal is to find the matrix $\boldsymbol{A}$ and the vector $\mathbf{b}$ such that

$$|\mathbf{y} - (\boldsymbol{A}\,\mathbf{x} + \mathbf{b})|^2 \tag{2.5}$$

is minimal for a chosen set of corresponding points $\mathbf{x} \in M_1$ and $\mathbf{y} \in M_2$. Let us have corresponding points $\mathbf{x_1}, \ldots, \mathbf{x_n}$ and $\mathbf{y_1}, \ldots, \mathbf{y_n}$, respectively. Such minimization for all corresponding points is equivalent to minimizing

$$\sum_{i=1}^{n} \left((y_{i_1} - a_1 x_{i_1} - a_2 x_{i_2} - b_1)^2 + (y_{i_2} - a_3 x_{i_1} - a_4 x_{i_2} - b_2)^2 \right) \tag{2.6}$$

with respect to the matrix and translation vector elements. In order to minimize (2.6) we compute the derivatives with respect to a_1, a_2, b_1, a_3, a_4 and b_2, and set it to 0.

For example, for the element a_1 we get

$$\sum_{i=1}^{n} 2 \left(y_{i_1} - a_1 x_{i_1} - a_2 x_{i_2} - b_1 \right) \left(-x_{i_1} \right) = 0 \qquad (2.7)$$

and similarly for other elements. Equation (2.7) can be written in the form

$$a_1 \sum_{i=1}^{n} x_{i_1} x_{i_1} + a_2 \sum_{i=1}^{n} x_{i_2} x_{i_1} + b_1 \sum_{i=1}^{n} x_{i_1} = \sum_{i=1}^{n} y_{i_1} x_{i_1} \qquad (2.8)$$

and from there we can see that for all elements we get the system of linear equations

$$\begin{bmatrix} \sum_{i=1}^{n} x_{i_1}^2 & \sum_{i=1}^{n} x_{i_1} x_{i_2} & \sum_{i=1}^{n} x_{i_1} & 0 & 0 & 0 \\ \sum_{i=1}^{n} x_{i_1} x_{i_2} & \sum_{i=1}^{n} x_{i_2}^2 & \sum_{i=1}^{n} x_{i_2} & 0 & 0 & 0 \\ \sum_{i=1}^{n} x_{i_1} & \sum_{i=1}^{n} x_{i_2} & n & 0 & 0 & 0 \\ 0 & 0 & 0 & \sum_{i=1}^{n} x_{i_1}^2 & \sum_{i=1}^{n} x_{i_1} x_{i_2} & \sum_{i=1}^{n} x_{i_1} \\ 0 & 0 & 0 & \sum_{i=1}^{n} x_{i_1} x_{i_2} & \sum_{i=1}^{n} x_{i_2}^2 & \sum_{i=1}^{n} x_{i_2} \\ 0 & 0 & 0 & \sum_{i=1}^{n} x_{i_1} & \sum_{i=1}^{n} x_{i_2} & n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ a_3 \\ a_4 \\ b_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_{i_1} x_{i_1} \\ \sum_{i=1}^{n} y_{i_1} x_{i_2} \\ \sum_{i=1}^{n} y_{i_1} \\ \sum_{i=1}^{n} y_{i_2} x_{i_1} \\ \sum_{i=1}^{n} y_{i_2} x_{i_2} \\ \sum_{i=1}^{n} y_{i_2} \end{bmatrix} \qquad (2.9)$$

Regarding our application to transform the *Strabo's map of the World* to the current map in the WGS, now we present the finding of affine transformations T^{p_k} by (2.9) for the sets of selected corresponding points, $p_k, k = 1, 2, 3$, with the corresponding points given at the Adriatic coast ($k = 1$), Greece and Albania region ($k = 2$) and at the Black Sea coast ($k = 3$), for more geographic information about the sets of corresponding points, see the beginning of Section 3. The accuracy of the affine transformation T^{p_k} for the set of corresponding points $p_j = \{\mathbf{x}_i, \mathbf{y}_i; i = 1, \ldots, n_{p_j}\}$ is measured by the mean error

$$\varepsilon_{\text{mean}}^{p_j} \left(T^{p_k} \right) = \sqrt{\frac{1}{n_{p_j}} \sum_{\mathbf{x}_i \in p_j} D_E \left(\mathbf{y}_i, T^{p_k} \left(\mathbf{x}_i \right) \right)^2}, \qquad (2.10)$$

where $D_E \left(\mathbf{y}_i, T^{p_k} \left(\mathbf{x}_i \right) \right)$ is a geodesic distance of point $\mathbf{y}_i$ and transformed point $T^{p_k} \left(\mathbf{x}_i \right)$ computed by the GeographicLib::Geodesic class [15]. We also measure the maximal error of the transformation by

$$\varepsilon_{\text{max}}^{p_j} \left(T^{p_k} \right) = \max_{\mathbf{x}_i \in p_j} D_E \left(\mathbf{y}_i, T^{p_k} \left(\mathbf{x}_i \right) \right). \qquad (2.11)$$

TABLE 2.1. The mean errors of the transformation T^{p_k} for the corresponding points sets p_j.

k	p_k - region	$\varepsilon_{\text{mean}}^{p_1}\left(T^{p_k}\right)$ [km]	$\varepsilon_{\text{mean}}^{p_2}\left(T^{p_k}\right)$ [km]	$\varepsilon_{\text{mean}}^{p_3}\left(T^{p_k}\right)$ [km]
1	Adriatic coast	7.527	256.856	723.619
2	Greece and Albania	197.655	24.259	126.697
3	Black Sea coast	155.672	189.790	22.981

TABLE 2.2. The maximal errors of the transformation T^{p_k} for the corresponding points sets p_j.

k	p_k - region	$\varepsilon_{\max}^{p_1}\left(T^{p_k}\right)$ [km]	$\varepsilon_{\max}^{p_2}\left(T^{p_k}\right)$ [km]	$\varepsilon_{\max}^{p_3}\left(T^{p_k}\right)$ [km]
1	Adriatic coast	12.201	383.310	917.010
2	Greece and Albania	235.104	32.042	178.451
3	Black Sea coast	198.112	232.691	35.989

TABLE 2.3. The mean and maximal errors of the transformation T^{pG} computed by using the global corresponding points set $p_G = p_1 \cup p_2 \cup p_3$. In the first three lines, the errors of T^{pG} for the corresponding points just from the sets $p_k, k = 1, 2, 3$ are presented. Comparing these errors with the errors on diagonals of Tables 2.1–2.2 we see the error increase. The last line represents the errors of T^{pG} for all corresponding points in the global set p_G.

k	p_k - region	$\varepsilon_{\text{mean}}^{p_k}\left(T^{pG}\right)$ [km]	$\varepsilon_{\max}^{p_k}\left(T^{pG}\right)$ [km]
1	Adriatic coast	45.730	65.453
2	Greece and Albania	62.224	92.509
3	Black Sea coast	43.519	79.303
G	Global	52.543	92.509

Transformations presented in the Tables 2.1–2.2 come with acceptable errors for the corresponding points sets p_k if the same corresponding points are used also for finding the transformation T^{p_k}, see diagonals of the tables. Unfortunately, the error dramatically rises for the corresponding points not used for finding the optimal transformation, see out of diagonal entries in the tables and Figures 2.1–2.3. Thus these local transformations are not suitable for transformation of farther points not included in the minimization procedure. Of course, one can create a common global corresponding points set

$$p_G = p_1 \cup p_2 \cup p_3$$

and find by (2.9) a common global transformation T^{pG}. Using such optimal affine transformation T^{pG} we get visually better results, see Fig 2.4. However, the errors $\varepsilon^{p_k}_{\mathrm{mean}}(T^{pG})$ and $\varepsilon^{p_k}_{\mathrm{max}}(T^{pG})$, $k = 1, 2, 3$, see Table 2.3, have significantly increased comparing to the corresponding errors on the diagonals of Tables 2.1--2.2. Also visually we see quite large differences compared to the local transformations. For example, for the Adriatic coast region the mean error rises from $7.5\,\mathrm{km}$ to $45.7\,\mathrm{km}$ and maximal error from $12.2\,\mathrm{km}$ to $65.4\,\mathrm{km}$ which is expressed visually in Figures 2.1 and 2.4. A similar result is observed near Istanbul and in other localities as well. Since we want to keep the accuracy of the local affine transformations in the neighbourhood of selected polygonal regions and not to pollute the transformations by the globally increasing errors we come with the following idea.

By using the points $\mathbf{x}_i \in p_k$ we create a polygon denoted again without any confusion by p_k, and it is done for all $k = 1, \ldots, n_p$ where n_p is the number of corresponding points sets and the created polygons as well. In all points of the map M_1, which are inside of every polygon p_k, we set the parameters of affine transformation to the values of locally optimal affine transformation T^{p_k} found by (2.9) using the corresponding points set p_k. For all other points of the map M_1 we use an interpolation/extrapolation approach. In 1D case, when we want to interpolate between two given function values, the natural approach is to use the linear interpolation, i.e., to connect two given values by a straight line. However, it is not so straightforward in higher dimensions. Fortunately, there exists an analogy of linear interpolation in higher dimensions given by a solution of the Laplace equation with Dirichlet boundary conditions given on the boundary of the interpolation domain. It is widely used in data processing such as filling the missing parts of photographs or other image inpainting problems. However, in our case, we do not have only to interpolate the values to the regions between the polygons but also to extrapolate these values to the whole map M_1. To that goal, we suggest the following mathematical model. Let us consider the Laplace equation

$$-\Delta u(\mathbf{x}) = 0, \qquad (2.12)$$

where solution u represents the transformation matrix and translation vector elements $a_1, a_2, b_1, a_3, a_4, b_2$, together with the Dirichlet conditions prescribed in the polygons p_k. The Laplace equation with such Dirichlet conditions is solved in a domain Ω and the zero Neumann boundary conditions are prescribed on its boundary $\partial\Omega$. The domain Ω is chosen as a rectangular subset of the map M_1, see, e.g., the pictures in Figure 2.7 where we choose as domain Ω a rectangle surrounding Europe on the Strabo's map of the World. We note that the minus sign in the equation (2.12) is chosen to have operator on the left hand side positive and arising matrix of the system then positive definite.
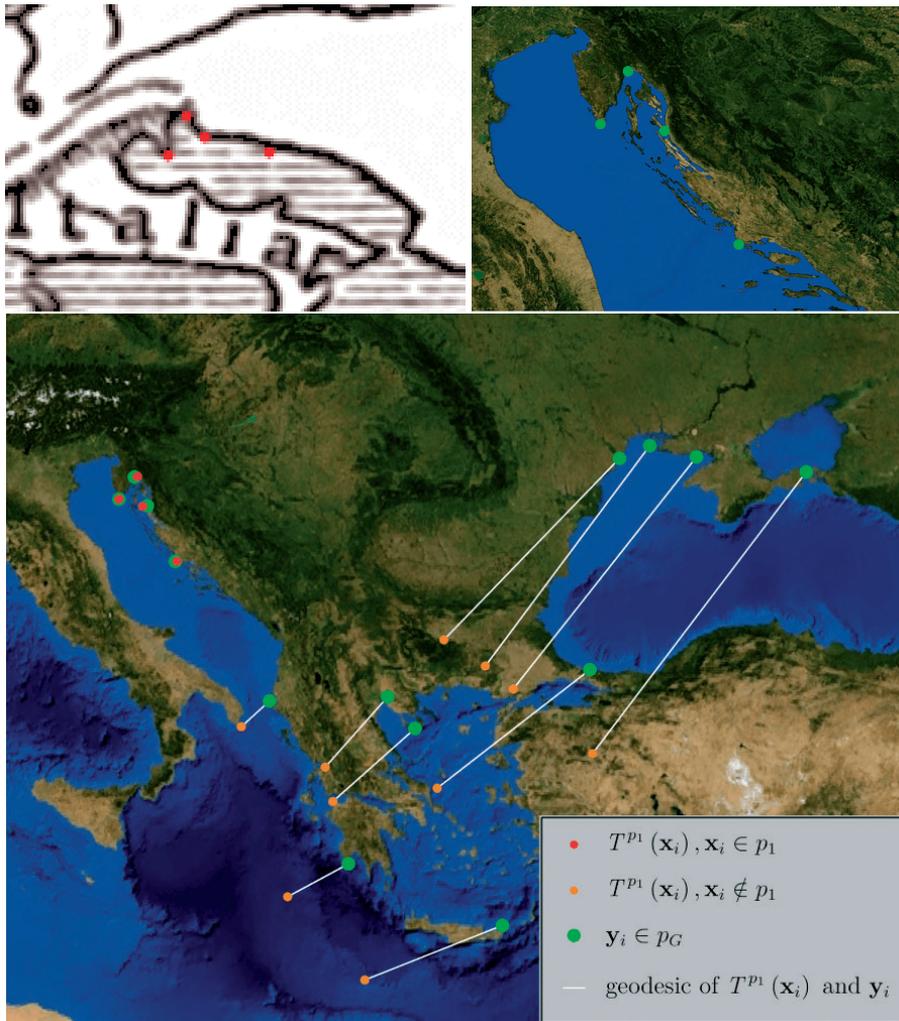
FIGURE 2.1. Optimal local affine transformation T^{p_1} obtained by using the corresponding points set p_1 given on the Adriatic coast. Top images show the corresponding points, on the left in red on the Strabo's map and on the right in green on the map in WGS. The bottom image shows the transformed points of the set p_1 in red, the transformed points of the sets p_2 and p_3 in orange and their corresponding points in WGS in green. One can see that points on Adriatic coast are transformed accurately while the points in other regions are distant from their corresponding points, see also the first rows of Tables 2.1–2.2.

FIGURE 2.2. Optimal local affine transformation T^{p_2} obtained by using the corresponding points set p_2 given in the Greece and Albania region. Top images show the corresponding points, on the left in red on the Strabo's map and on the right in green on the map in WGS. The bottom image shows the transformed points of the set p_2 in red, the transformed points of the sets p_1 and p_3 in orange and their corresponding points in WGS in green. One can see that points in the Greece and Albania region are transformed accurately while the points in other regions are distant from their corresponding points, see also the second rows of Tables 2.1–2.2.

FIGURE 2.3. Optimal local affine transformation T^{p_3} obtained by using the corresponding points set p_3 given on the Black Sea coast. Top images show the corresponding points, on the left in red on the Strabo's map and on the right in green on the map in WGS. The bottom image shows the transformed points of the set p_3 in red, the transformed points of the sets p_1 and p_2 in orange and their corresponding points in WGS in green. One can see that points on Black Sea coast are transformed accurately while the points in other regions are distant from their corresponding points, see also the third rows of Tables 2.1–2.2.
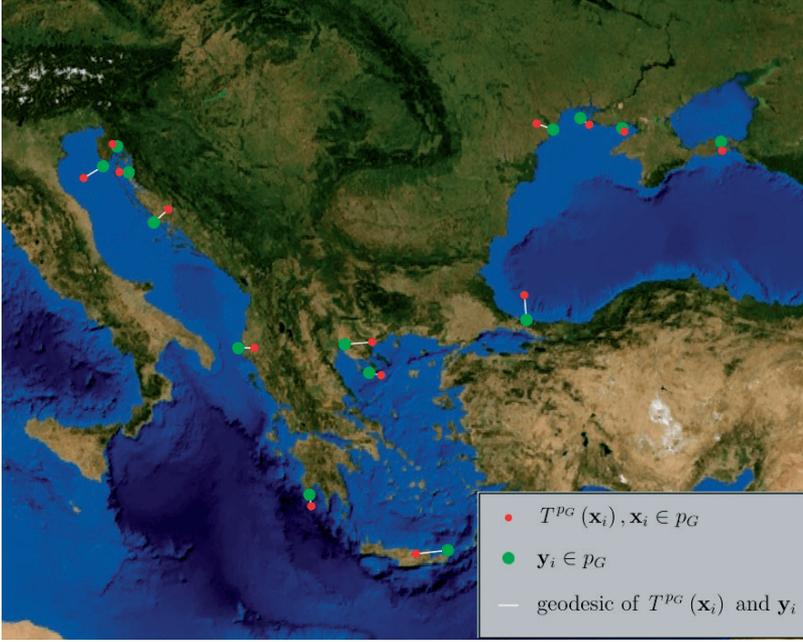
FIGURE 2.4. Optimal local affine transformation T^{p_G} obtained by using the corresponding points set $p_G = p_1 \cup p_2 \cup p_3$. The image shows the transformed points from the Strabo's map in red and their corresponding points in WGS in green. One can see that all the points are transformed relatively closely. However, in the previous local transformations, see Figures 2.1–2.3, the points from the sets used for finding the optimal local affine transformations were transformed more accurately, see also Tables 2.1–2.3.

To discretize the partial differential equation (2.12) in the domain Ω we use the finite difference method on a uniform grid with the grid size 1, see Fig. 2.5. The grid nodes $\mathbf{x}_{i,j}, i = 1, \ldots, N_1, j = 1, \ldots, N_2$ correspond to centers of pixels in the map M_1. The Dirichlet conditions are prescribed in $E(p_k)$, the outer discrete envelope of the polygons p_k, see Fig. 2.5 case A. In other grid nodes except the boundary, case B in Fig. 2.5, the Laplace operator in equation (2.12) is approximated by

$$\Delta u\left(\mathbf{x}\right) = \frac{\partial^2 u}{\partial x_1{}^2}\left(\mathbf{x}\right) + \frac{\partial^2 u}{\partial x_2{}^2}\left(\mathbf{x}\right) \tag{2.13}$$

$$\approx u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + u_{i,j-1} - 2u_{i,j} + u_{i,j+1} \tag{2.14}$$

$$= u_{i-1,j} + u_{i+1,j} - 4u_{i,j} + u_{i,j-1} + u_{i,j+1}, \tag{2.15}$$

where $u_{i,j}$ is an approximate value of u at the grid node $\mathbf{x}_{i,j}$.

Such an approximation needs to be adjusted for the grid points on the boundary $\partial\Omega$. To approximate the zero Neumann boundary condition we use the reflection of values along the boundary, e.g., in case J in Fig. 2.5 we set $u_{i-1,j} = u_{i+1,j}$ which leads to the following approximation on that part of the boundary

$$\Delta u\,(\mathbf{x}) \approx u_{i+1,j} - 2u_{i,j} + u_{i+1,j} + u_{i,j-1} - 2u_{i,j} + u_{i,j+1} \tag{2.16}$$

$$= 2u_{i+1,j} - 4u_{i,j} + u_{i,j-1} + u_{i,j+1} \tag{2.17}$$

and it is done similarly for other boundary grid nodes.

We summarize all discrete equations, for cases A–J, representing the numerical discretization of our model for the LAGL map transformation as follows:

A: $\mathbf{x}_{i,j} \in E(p_k) : u_{i,j} = v\,(p_k)$,

where v is any of the element of $\boldsymbol{A}$ and $\mathbf{b}$ given by locally optimal affine transformation T^{p_k} found by (2.9),

B: $\mathbf{x}_{i,j} \notin E(p_k) \wedge \mathbf{x}_{i,j} \notin \partial\Omega : -u_{i-1,j} - u_{i,j-1} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1} = 0$,

C: $\mathbf{x}_{i,j} \notin E(p_k) \wedge i = 1, j = 1 : 4u_{i,j} - 2u_{i+1,j} - 2u_{i,j+1} = 0$,

D: $\mathbf{x}_{i,j} \notin E(p_k) \wedge i = 1, j = N_2 : -2u_{i,j-1} + 4u_{i,j} - 2u_{i+1,j} = 0$,

E: $\mathbf{x}_{i,j} \notin E(p_k) \wedge i = N_1, j = 1 : -2u_{i-1,j} + 4u_{i,j} - 2u_{i,j+1} = 0$,

F: $\mathbf{x}_{i,j} \notin E(p_k) \wedge i = N_1, j = N_2 : -2u_{i-1,j} - 2u_{i,j-1} + 4u_{i,j} = 0$,

G: $\mathbf{x}_{i,j} \notin E(p_k) \wedge i = N_1, j = 2, \ldots, N_2-1 : -2u_{i-1,j} - u_{i,j-1} + 4u_{i,j} - u_{i,j+1} = 0$,

H: $\mathbf{x}_{i,j} \notin E(p_k) \wedge i = 2, \ldots, N_1-1, j = N_2 : -u_{i-1,j} - 2u_{i,j-1} + 4u_{i,j} - u_{i+1,j} = 0$,

I: $\mathbf{x}_{i,j} \notin E(p_k) \wedge i = 2, \ldots, N_1-1, j = 1 : -u_{i-1,j} + 4u_{i,j} - u_{i+1,j} - 2u_{i,j+1} = 0$,

J: $\mathbf{x}_{i,j} \notin E(p_k) \wedge i = 1, j = 2, \ldots, N_2-1 : -u_{i,j-1} + 4u_{i,j} - 2u_{i+1,j} - u_{i,j+1} = 0$.

For solving the above system of equations we use Eigen::SparseLU class of [11] which solves the linear systems with sparse matrices directly and efficiently by the LU decomposition. Solution of the linear system of equations gives us the desired result, smoothly interpolated/extrapolated locally optimal affine transformations from polygons p_k to every point of the (historical) map M_1. As we see in Figure 2.6, we reproduce the optimal errors (diagonals in Tables 2.1–2.2) of all local affine transformations thanks to the Dirichlet conditions in polygons p_k. These optimal values are smoothly interpolated/extrapolated to the whole historical map M_1, as seen in Fig. 2.7. Such locally optimal smoothly varying transformation can be used to transform any point from the historical map M_1 to the current map M_2.
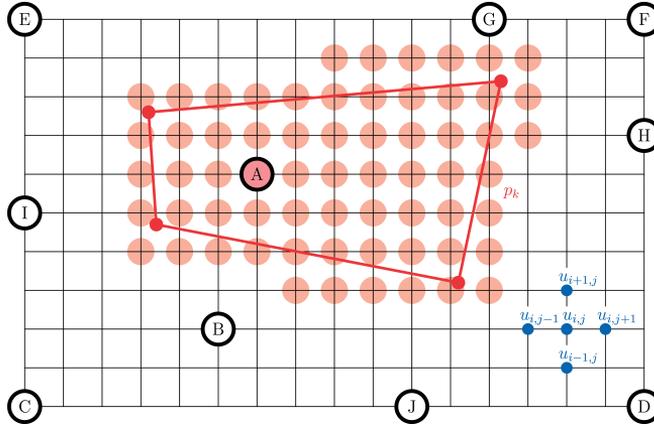
FIGURE 2.5. Illustration of the discretization of the computational domain Ω and approximation of our mathematical model in cases $A - J$. By pink color we plot the grid nodes in the outer discrete envelope $E(p_k)$ of the polygon p_k (plotted in red), where we consider Dirichlet conditions. In case B we consider the standard approximation of the Laplace equation while in cases $C - J$ we consider its adjustment at the boundary $\partial\Omega$.
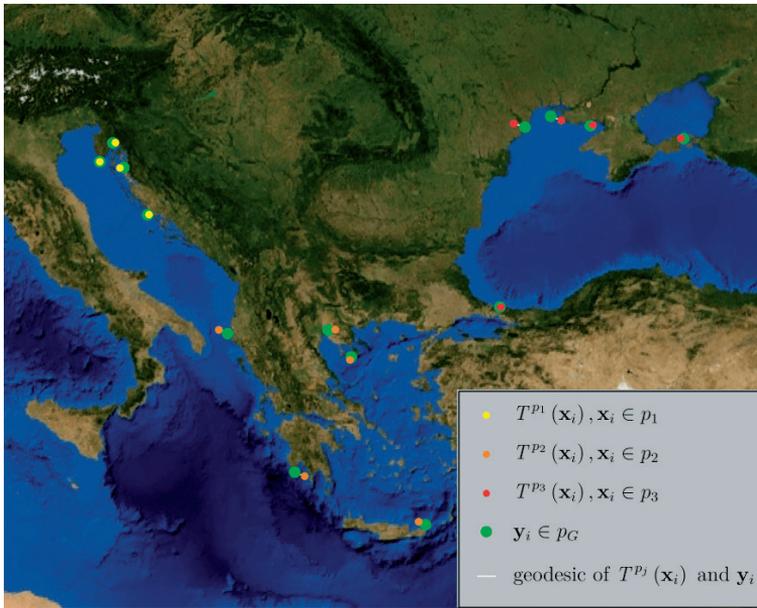
FIGURE 2.6. Transformation found by using the LAGL method. The image illustrates the result for the corresponding points sets p_1, p_2 and p_3 which are transformed as accurately as by their locally optimal affine transformations T^{p_1}, T^{p_2} and T^{p_3}, see Figures 2.1–2.3.
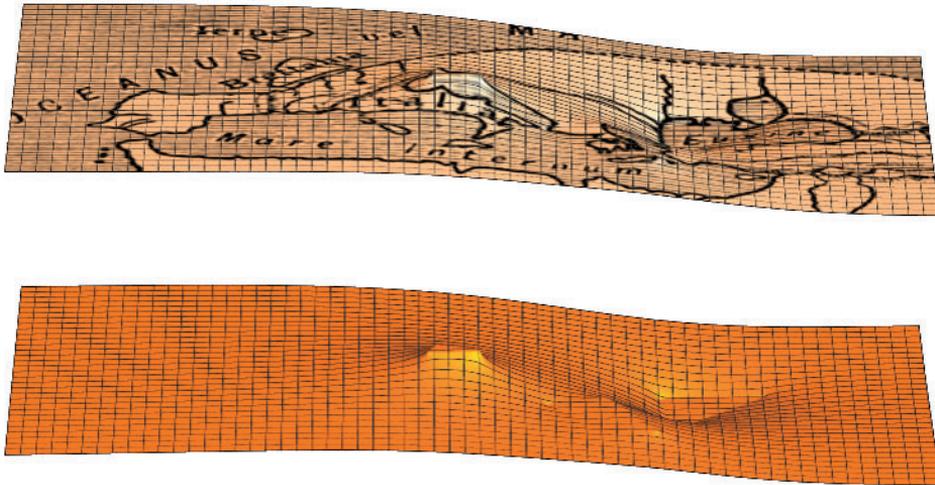
FIGURE 2.7. The images show one of the transformation parameters computed by the LAGL method presented in Figure 2.6. The upper image is plotted with the texture of the *Strabo's map of the World*, the lower image is plotted without the texture emphasizing the smooth transition of locally optimal affine transformations.

3. Strabo's Ister (Ἴστρος) transformation

Now we present the application of the LAGL method to the transformation of the river Ister from the *Strabo's map of the World* to the current map in WGS. The corresponding points on Strabo's map are chosen on the east, south and west directions from the Ister river on the *Strabo's map of the World* and in such a way that they are reliably identifiable also on the current map of the world. It is clearly observable that the *Strabo's map of the World* by Karl Müller is less realistic in the north-western and northern directions from the Alps than in other directions. But this fact correctly respects the uncertainty of Strabo's description of this part of Europe and Karl Müller's map does not introduce any artificial information which can be found in others 19th-century map reconstructions of the *Geographica*. While southern Europe, west of the Alps up to Lyon (Λούγδουνον) and east of the Alps up to the Ister, is described in *Geographica* with a lot of quantitative geographic details including distances, the northern part contains almost no such information. The almost only quantitative information is what Strabo says about the length of the Rhine: "Asinius (historian) says that length of the river (Rhine) is six thousand stadia, but it is not, for it would be only a little more than half that in a straight line, and adding a thousand would be sufficient for bending.", see [30], Section (4–3–3). The estimate of six

thousand stadia (by Asinius) is however a right one since the current length of the Rhine is about 1230 km, but Strabo does not accept that and almost halved it. This shows how uncertain was the knowledge about the river Rhine on the northern planes in the time of Strabo and due to that also on the Karl Müller's map it is much shorter than in the reality. About the northern part of Europe Strabo also writes that the region "near the Ocean is totally unknown to us", see [30], Section (7–2–4), and by the "Ocean", he means the nowadays Northern and Baltic Seas. That is the reason why Karl Muller plotted the northern border of Europe (and of the World) by the dashed line. Due to the above reasons, we are not allowed to choose the corresponding points north-west and north of the Ister.

Having all these requirements in mind, we defined the first three sets of corresponding points on the Mediterranean and the Black Sea costs - in the south- -west, south and east directions from the Strabo's Ister, see Figures 2.1–2.4 and Figure 3.2.1. Since the Mediterranean world up to the Ister river and along it as well as the regions along the Black Sea were already well-known for Greeks and Romans in the time of Strabo, the chosen points are reliable regarding location, distances and directions. The first three corresponding points sets defined below fulfils all the above assumptions, but we created also the fourth one containing the points on the Italian coastline and in the Alps. The points in the Alps - the sources of Rhone (Ῥοδανὸς) and Rhine rivers seen on the Strabo's map - have some uncertainty to be identified correctly on the current map. We put the "source" of the Rhine below the Rheinwaldhorn, since in section (4–6–6) Strabo writes that it flows from the mount Adula (Ἀδούλας), and the "source" of Rhona to the Lake Geneva (lac Léman) because in (4–6–6) Strabo writes that it bears from Λημέννα λίμνη. We use this fourth corresponding points set mainly to show that considering the western Alpine region, although a bit uncertain, does not change the results significantly. At this place, we would also like to note (probably the only one) discrepancy of the Karl Müller's map and Strabo's descriptions in *Geographica*. It is the sketch of the Alps, namely, the "second ridge" should encompass the sources of all the Ister, Rhein and the Rhone because all these river sources are within the Alps by Strabo. We think that Karl Müller's sketch of the "second ridge" is related to the 19th–20th-century standard assumption of the correspondence of the Strabo's Ister and nowadays Danube rivers.

Here are the corresponding points sets:

Adriatic coast region:

- south of Istria (Premantura),
- Opatija,
- Jablanac,
- Split area (Ražanj).

Black Sea coast region:

- Istanbul (north of Bosporus),
- mouth of Dniester (Zatoka),
- cape of Tendrivska gulf,
- cape of Dzharylhatska gulf,
- Kerč.

Greece and Albania region:

- Vlorë,
- Koufasaratsia
- north-east cape of Crete (Kyriamadi),
- cape of Kassandra peninsula,
- Thessaloniki.

Alps and Italy region:

- Ancona (Conero),
- cape south of Venice,
- Trieste,
- "source" of Rhine (Rheinwaldhorn),
- "source" of Rhone (Lake Geneva).

In numerical experiments presented in this section we vary following combinations of the above-defined regions:

Experiment 3.1: Adriatic coast, Black Sea coast regions;

Experiment 3.2: Adriatic coast, Greece and Albania, Black Sea coast regions;

Experiment 3.3: Adriatic coast, Black Sea coast, Alps and Italy regions.

We will use the following abbreviations:

D : Danube;

D1 : the Danube from the source up to the confluence with the Drava;

D2 : the Danube from the confluence with the Drava up to outlet to the Black Sea;

DD : courses of Drava and Danube interconnected;

TID : courses of Tauernbach, Isel and Drava up to the confluence with the Danube;

TIDD : courses of Tauernbach, Isel, Drava and Danube interconnected;

I1 : transformed Ister from the source up to the intersection with the Danube;

I2 : transformed Ister from the intersection with Danube up to outlets to the Black Sea.

To evaluate the results of transformations we compute the maximal and the mean Hausdorff distances (defined below) of the two discrete curves - one representing the real river course, precisely digitized on the current map, and one representing the discrete transformed Ister from the Strabo's map to the current map. To get the transformed Ister, first, the Ister on the Strabo's map, see Figure 1.1, was digitized to contiguous pixel set and then every center of the pixel was transformed to the current map by the LAGL map transformation. All necessary distances on the current map in WGS are computed by means of the GeographicLib::Geodesic class [15]. In Figures 3.1.1–3.3.2, the cyan curve represents always the transformed Strabo's Ister while the white curve represents the Danube river, yellow curve the Drava river, the orange curve represents the Isel river and red curve the Tauernbach. We also measure the length of two curves matching in a prescribed narrow band by the so-called matching length defined below.

Let us have a discrete curve

$$A = \{\mathbf{a}_1, \ldots, \mathbf{a}_{n_A}\}.$$

By using the points

$$\mathbf{a}_i, i = 1, \ldots, n_A$$

we create the piecewise linear segments

$$\widehat{A} = \{\widehat{\mathbf{a}}_1, \ldots, \widehat{\mathbf{a}}_{n_A}\}$$

as follows

$$\widehat{\mathbf{a}}_1 = \overline{\mathbf{a}_1, \frac{\mathbf{a}_1 + \mathbf{a}_2}{2}}, \tag{3.1}$$

$$\widehat{\mathbf{a}}_i = \overline{\frac{\mathbf{a}_{i-1} + \mathbf{a}_i}{2}, \mathbf{a}_i} \cup \overline{\mathbf{a}_i, \frac{\mathbf{a}_i + \mathbf{a}_{i+1}}{2}}, \quad i = 2, \ldots, n_A - 1, \tag{3.2}$$

$$\widehat{\mathbf{a}}_{n_A} = \overline{\frac{\mathbf{a}_{n_A-1} + \mathbf{a}_{n_A}}{2}, \mathbf{a}_{n_A}}, \tag{3.3}$$

where $\overline{\mathbf{u}, \mathbf{v}}$ represents the line segment connecting points $\mathbf{u}$ and $\mathbf{v}$. Let $\widehat{a}_i = |\widehat{\mathbf{a}}_i|$ be the piecewise linear segment length and let L_A be the length of the overall discrete curve A given by the sum of the segments length.

The so-called directed mean Hausdorff distance $\overline{\delta}_H(A, B)$ of two discrete curves

$$A = \{\mathbf{a}_1, \ldots, \mathbf{a}_{n_A}\} \quad \text{and} \quad B = \{\mathbf{b}_1, \ldots, \mathbf{b}_{n_B}\}$$

with segments

$$\widehat{A} = \{\widehat{\mathbf{a}}_1, \ldots, \widehat{\mathbf{a}}_{n_A}\} \quad \text{and} \quad \widehat{B} = \left\{\widehat{\mathbf{b}}_1, \ldots, \widehat{\mathbf{b}}_{n_B}\right\}$$

is given by

$$\overline{\delta}_H(A, B) = \frac{1}{L_A} \sum_{i=1}^{n_A} \widehat{a}_i \min_{\widehat{\mathbf{b}}_j \in \widehat{B}} D_E(\mathbf{a}_i, \widehat{\mathbf{b}}_j), \tag{3.4}$$

where $D_E(\mathbf{a}_i, \widehat{\mathbf{b}}_j)$ is the geodesic distance of the point $\mathbf{a}_i$ and the segment $\widehat{\mathbf{b}}_j \in \widehat{B}$.

Then the **mean Hausdorff distance** $\overline{\mathrm{d}}_H(A, B)$ is given by the following formula

$$\overline{\mathrm{d}}_H(A, B) = \frac{L_A \overline{\delta}_H(A, B) + L_B \overline{\delta}_H(B, A)}{L_A + L_B}. \tag{3.5}$$

The so-called directed maximal Hausdorff distance $\delta_H(A, B)$ is given by

$$\delta_H(A, B) = \sup_{\mathbf{a}_i \in A} \inf_{\widehat{\mathbf{b}}_j \in \widehat{B}} D_E(\mathbf{a}_i, \widehat{\mathbf{b}}_j) \tag{3.6}$$

and then the **maximal Hausdorff distance** $\mathrm{d}_H(A, B)$ is given by

$$\mathrm{d}_H(A, B) = \max \left\{ \delta_H(A, B), \delta_H(B, A) \right\}. \tag{3.7}$$

Finally, the sum of length of segments $\widehat{a}_i \in \widehat{A}$ within the given threshold distance d_t from the segments of the curve B gives the **matching length** $L_m(A, B, d_t)$ by

$$L_m(A, B, d_t) = \sum_{\widehat{\mathbf{a}}_i \in \widehat{A}} \widehat{a}_i, \tag{3.8}$$

$$\min_{\widehat{\mathbf{b}}_j \in \widehat{B}} D_E(\mathbf{a}_i, \widehat{\mathbf{b}}_j) < d_t. \tag{3.9}$$

In the following Figures and Tables, we present the results of Experiments 3.1--3.3 representing three different LAGL transformations of the Strabo's Ister to the current map in WGS. In Figures, we evaluate the results visually and in Tables quantitatively.

In Figures 3.x.1 ($x = 1, 2, 3$) we visualize the polygons used for finding the locally optimal affine transformations used in LAGL method, the transformed river Ister (cyan) and the rivers Danube, Drava, Isel and Tauernbach (various colors). In all these Figures the Ister in its upper course is really close to the Drava/Tauernbach-Isel-Drava(TID) river courses.

In Figures 3.x.2 we compare visually the river sources with the source of the transformed Ister. We see that all sources of Drava, Isel and Tauernbach are geographically very close to the source of the transformed Ister.

Tables 3.x.1 show that the sources of Danube and transformed Ister are very distant, around 300 km in all transformations, while the sources of Drava, Isel and Tauernbach are all much closer to the source of transformed Ister in all transformations, e.g., in Table 3.1.1 all distances are in the range from around 20 km to around 45 km, and they are slightly bigger in other Tables.

Now, let us look to Tables 3.x.2. Both the maximal and the mean Hausdorff distances (HD) are bigger when comparing Danube and Ister than when comparing Ister to the other river courses in their full length, see the first part of the Tables (first three rows). This difference is significantly emphasized in the second part of the Tables (fourth to the sixth row), where only the partial upper river courses are compared. The Hausdorff distances of Danube river and the reconstructed Ister on the upper part of their courses (HD of D1 and I1) are really high - the maximal HD is about 300 km and the mean HD is about 150 km. Opposite to that fact, the maximal and the mean HD of the transformed Ister and Drava/Tauernbach-Isel-Drava (TID) course are much lower. For example, in Table 3.1.2 the maximal Hausdorff distances are about 40 km and the mean Hausdorff distances are about 20 km only, which quantitatively express the visual similarity of the transformed Ister and Drava/TID courses in Figure 3.1.1.

Tables 3.x.3 show the matching lengths in three different narrow bands 10, 50 and 100 km. It is another way to show how closely are the river streams on their length. The longer common length in a narrow band the better correspondence of the river courses is detected. As we see again in the second parts of these Tables (fifth to the sixth row), the matching length of upper courses of the transformed Ister and Drava/TID rivers is very high in narrow band 100 km (close or equal to 100 %) for all three LAGL transformations. The matching length is also very high in 50 km narrow band for the first two experiments and there is a similarity of river courses also in 10 km narrow band in the first experiment, which again show the perfect correspondence of the upper Ister and Drava/TID courses. On the other hand, there is almost no similarity of the transformed Ister and the Danube river in its upper course as seen in the fourth row of the Tables.

The third part (the seventh row) of Tables 3.x.2 and 3.x.3 evaluate the quality of LAGL transformation of the Ister river. Since there is no doubt that the lower course of Danube and the transformed Ister should correspond to each other, the transformation which gives the lowest Hausdorff distances and the highest matching length on these partial river courses is the most reliable concerning the accuracy of the Strabo's Ister reconstruction. As one can see, from this point of view the most accurate is the LAGLtransformation from Experiment 3.1, using just the Adriatic and the Black Sea coast regions. From the above discussion, we see that it also gives the best Strabo's Ister and Tauerbnbach-Isel-Drava-Danube correspondence.
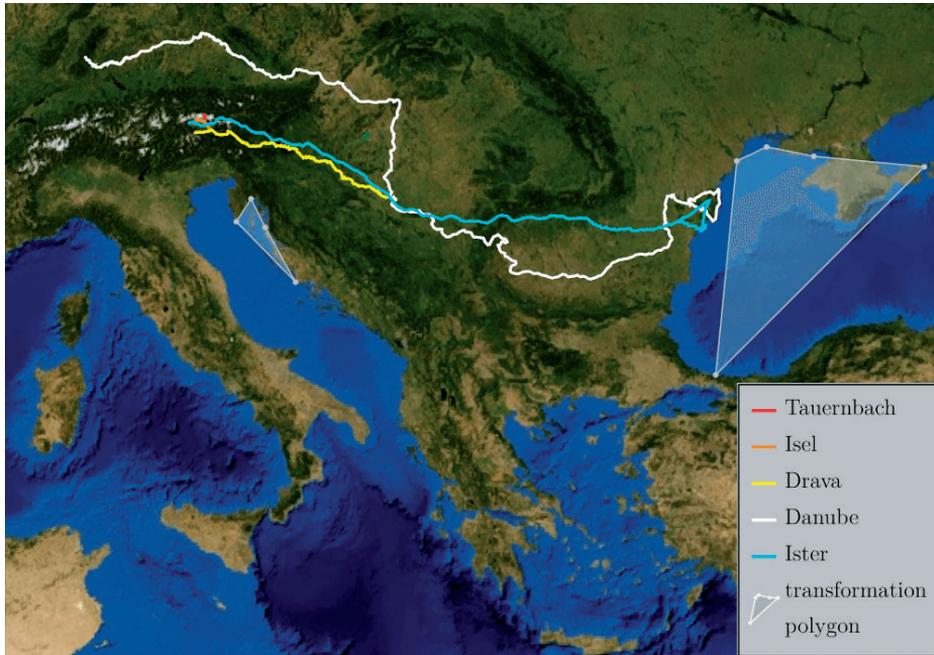
FIGURE 3.1.1. Visual comparison of current rivers (various colors) and river Ister (cyan) transformed using Adriatic and Black Sea coast regions, the polygons used for the transformations are highlighted in grey.
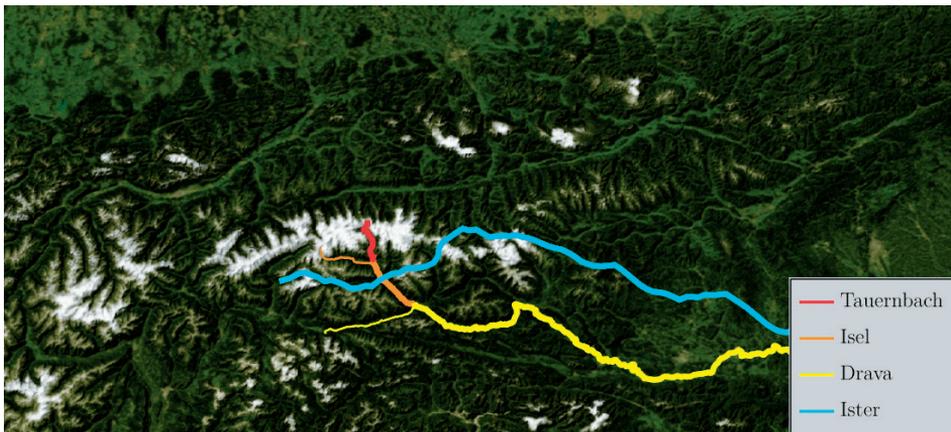
FIGURE 3.1.2. Detail of real rivers source (various colors) compared to the source of transformed river Ister (cyan). We plot the course Tauernabach-Isel-Drava thicker since we consider it to be Ister.

TABLE 3.1.1. The distance of river sources.

Danube - Ister	Drava - Ister	Isel - Ister	Tauernbach - Ister
317.781 km	29.849 km	21.776 km	45.109 km

TABLE 3.1.2. The table contains three parts. In the first part, the Hausdorff distances (HD) of the river courses in their full length are presented. Then the partial river courses are compared.

Curve		Maximal HD [km]		Mean HD [km]	
A	B	$\delta_H(A,B)$	$d_H(A,B)$	$\overline{\delta}_H(A,B)$	$\overline{d}_H(A,B)$
D	Ister	317.781		112.802	
Ister	D	206.829	317.781	79.573	100.898
DD	Ister	154.611		44.903	
Ister	DD	149.693	154.611	39.621	42.686
TIDD	Ister	154.611		44.717	
Ister	TIDD	149.693	154.611	39.484	42.524
D1	I1	317.781		177.243	
I1	D1	206.829	317.781	130.218	163.057
Drava	I1	39.020		21.831	
I1	Drava	38.809	39.020	20.547	21.246
TID	I1	39.020		21.383	
I1	TID	40.628	40.628	20.173	20.833
D2	I2	154.611		65.344	
I2	D2	149.693	154.611	59.130	62.953

TABLE 3.1.3. The table contains three parts. In the first part, the matching lengths of the river courses in their full length are presented. Then the partial river courses are compared.

Curve			Matching length $L_m(A, B, d_t)$ [km]		
A	L_A [km]	B	$d_t = 10$	$d_t = 50$	$d_t = 100$
D	2714.887	Ister	228.127 (8.4 %)	633.263 (23.3 %)	1114.154 (41.0 %)
Ister	1421.117	D	185.952 (13.0 %)	493.414 (34.7 %)	782.470 (55.0 %)
	A v e r a g e		207.040 (10.0 %)	563.339 (27.2 %)	948.312 (45.8 %)
DD	2021.995	Ister	299.190 (14.7 %)	1285.049 (63.5 %)	1693.837 (83.7 %)
Ister	1421.117	DD	267.783 (18.8 %)	1011.383 (71.1 %)	1247.252 (87.7 %)
	A v e r a g e		283.486 (16.4 %)	1148.216 (66.6 %)	1470.545 (85.4 %)
TIDD	2026.547	Ister	320.930 (15.8 %)	1289.600 (63.6 %)	1698.389 (83.8 %)
Ister	1421.117	TIDD	287.667 (20.2 %)	1011.383 (71.1 %)	1247.252 (87.7 %)
	A v e r a g e		304.299 (17.6 %)	1150.492 (66.7 %)	1472.821 (85.4 %)
D1	1430.501	I1	16.857 (1.1 %)	85.824 (5.9 %)	157.926 (11.0 %)
I1	617.966	D1	24.146 (3.9 %)	99.998 (16.1 %)	153.185 (24.7 %)
	A v e r a g e		20.501 (2.0 %)	92.911 (9.0 %)	155.555 (15.1 %)
Drava	737.609	I1	87.920 (11.9 %)	737.609 (100.0 %)	737.609 (100.0 %)
I1	617.966	Drava	105.976 (17.1 %)	617.966 (100.0 %)	617.966 (100.0 %)
	A v e r a g e		96.948 (14.3 %)	677.788 (100.0 %)	677.788 (100.0 %)
TID	742.161	I1	109.660 (14.7 %)	742.161 (100.0 %)	742.161 (100.0 %)
I1	617.966	TID	125.860 (20.3 %)	617.966 (100.0 %)	617.966 (100.0 %)
	A v e r a g e		117.760 (17.3 %)	680.063 (100.0 %)	680.063 (100.0 %)
D2	1284.386	I2	211.270 (16.4 %)	547.439 (42.6 %)	956.228 (74.4 %)
I2	803.151	D2	161.806 (20.1 %)	393.416 (48.9 %)	629.285 (78.3 %)
	A v e r a g e		186.538 (17.8 %)	470.428 (45.0 %)	792.757 (75.9 %)
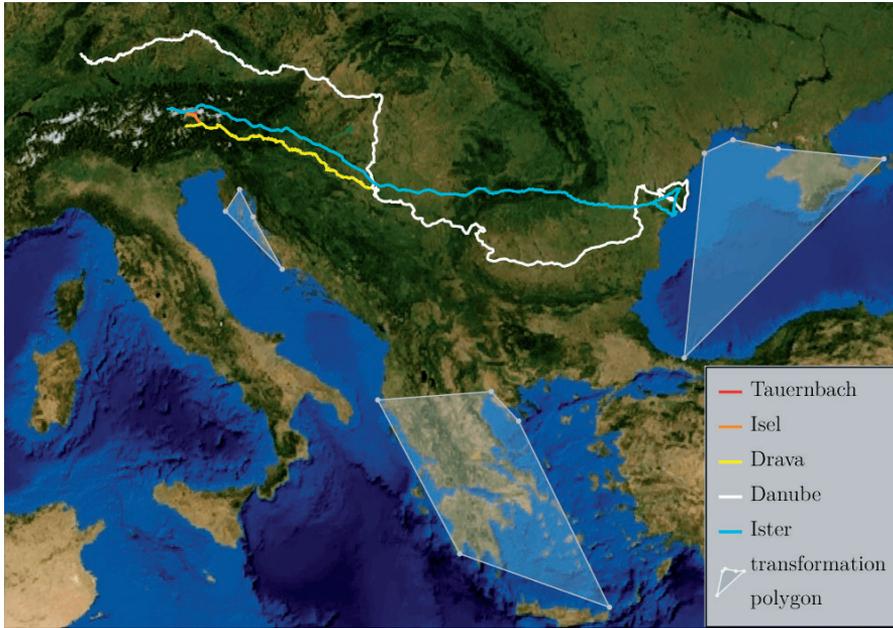
FIGURE 3.2.1. Visual comparison of current rivers (various colors) and river Ister (cyan) transformed using Adriatic, Black Sea and Greece and Albania regions, the polygons used for the transformations are highlighted in grey.
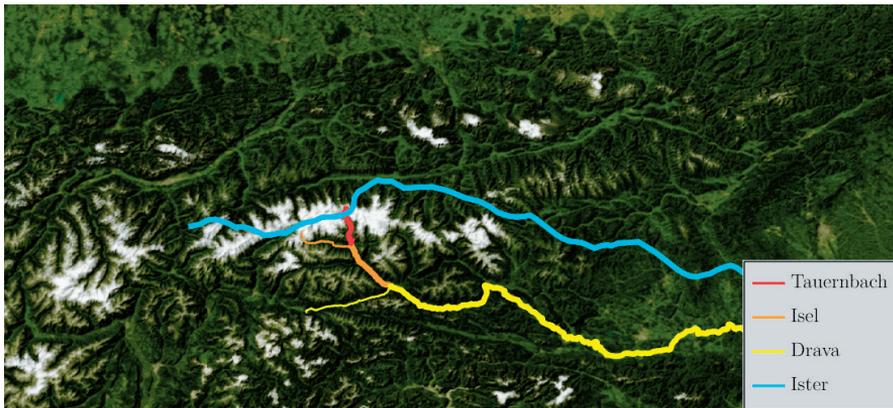
FIGURE 3.2.2. Detail of real rivers source (various colors) compared to the source of transformed river Ister (cyan). We plot the course Tauernabach-Isel-Drava thicker since we consider it to be Ister.

TABLE 3.2.1. The distance of river sources.

Danube - Ister	Drava - Ister	Isel - Ister	Tauernbach - Ister
278.962 km	67.198 km	51.800 km	72.642 km

TABLE 3.2.2. The table contains three parts. In the first part, the Hausdorff distances (HD) of the river courses in their full length are presented. Then the partial river courses are compared.

Curve(river)		Maximal HD [km]		Mean HD [km]	
A	B	$\delta_H(A,B)$	$d_H(A,B)$	$\overline{\delta}_H(A,B)$	$\overline{d}_H(A,B)$
D	Ister	278.962		116.695	
Ister	D	194.446	278.962	92.584	108.088
DD	Ister	198.848		66.131	
Ister	DD	189.915	198.848	57.998	62.729
TIDD	Ister	198.848		65.625	
Ister	TIDD	189.915	198.848	56.683	61.889
D1	I1	278.962		157.633	
I1	D1	194.446	278.962	125.898	147.751
Drava	I1	54.656		35.270	
I1	Drava	66.735	66.735	35.623	35.435
TID	I1	54.656		33.857	
I1	TID	70.863	70.863	32.194	33.083
D2	I2	198.848		96.661	
I2	D2	189.915	198.848	88.555	93.564

Table 3.2.3. The table contains three parts. In the first part, the matching lengths of the river courses in their full length are presented. Then the partial river courses are compared.

| Curve(river)			Matching length $L_m(A, B, d_t)$ [km]					
A	L_A [km]	B	$d_t = 10$		$d_t = 50$		$d_t = 100$	
D	2714.887	Ister	61.260	(2.2 %)	481.784	(17.7 %)	948.962	(34.9 %)
Ister	1441.085	D	55.447	(3.8 %)	323.687	(22.4 %)	641.770	(44.5 %)
	A v e r a g e		58.353	(2.8 %)	402.736	(19.3 %)	795.366	(38.2 %)
DD	2021.995	Ister	26.097	(1.2 %)	1082.523	(53.5 %)	1508.358	(74.5 %)
Ister	1441.085	DD	26.768	(1.8 %)	852.692	(59.1 %)	1144.834	(79.4 %)
	A v e r a g e		26.433	(1.5 %)	967.608	(55.8 %)	1326.596	(76.6 %)
TIDD	2026.547	Ister	41.406	(2.0 %)	1087.075	(53.6 %)	1512.910	(74.6 %)
Ister	1441.085	TIDD	50.989	(3.5 %)	855.911	(59.3 %)	1144.834	(79.4 %)
	A v e r a g e		46.197	(2.6 %)	971.493	(56.0 %)	1328.872	(76.6 %)
D1	1430.501	I1	34.437	(2.4 %)	112.681	(7.8 %)	178.212	(12.4 %)
I1	646.869	D1	25.465	(3.9 %)	90.754	(14.0 %)	143.805	(22.2 %)
	A v e r a g e		29.951	(2.8 %)	101.717	(9.7 %)	161.008	(15.5 %)
Drava	737.609	I1	0.000	(0.0 %)	713.419	(96.7 %)	737.609	(100.0 %)
I1	646.869	Drava	0.000	(0.0 %)	619.759	(95.8 %)	646.869	(100.0 %)
	A v e r a g e		0.000	(0.0 %)	666.589	(96.2 %)	692.239	(100.0 %)
TID	742.161	I1	15.308	(2.0 %)	717.971	(96.7 %)	742.161	(100.0 %)
I1	646.869	TID	24.221	(3.7 %)	622.978	(96.3 %)	646.869	(100.0 %)
	A v e r a g e		19.764	(2.8 %)	670.474	(96.5 %)	694.515	(100.0 %)
D2	1284.386	I2	26.097	(2.0 %)	369.103	(28.7 %)	770.749	(60.0 %)
I2	794.215	D2	26.768	(3.3 %)	232.933	(29.3 %)	497.964	(62.6 %)
	A v e r a g e		26.433	(2.5 %)	301.018	(28.9 %)	634.357	(61.0 %)
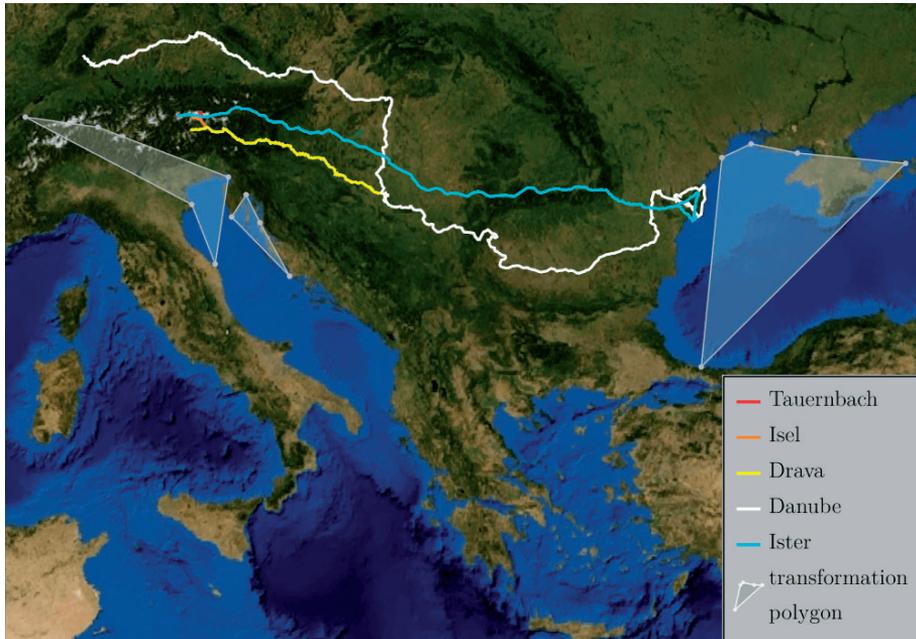
FIGURE 3.3.1. Visual comparison of current rivers (various colors) and river Ister (cyan) transformed using Adriatic, Black Sea and Alps and Italy regions, the polygons used for the transformations are highlighted in grey.
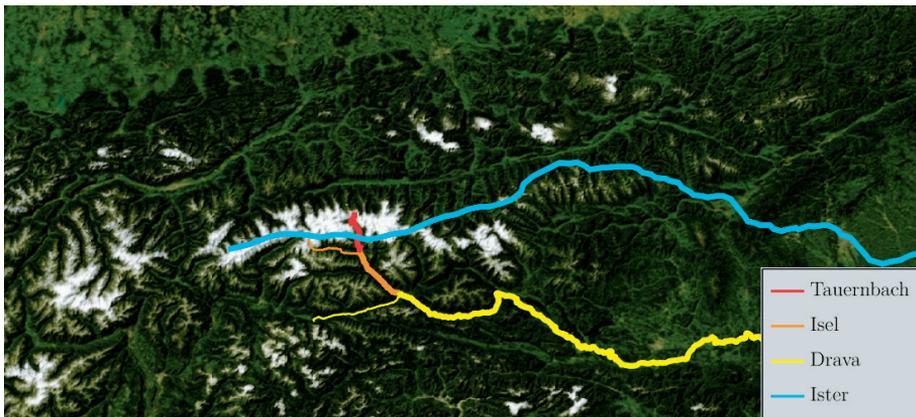
FIGURE 3.3.2. Detail of real rivers source (various colors) compared to the source of transformed river Ister (cyan). We plot the course Tauernabach--Isel-Drava thicker since we consider it to be Ister.

TABLE 3.3.1. The distance of river sources.

Danube - Ister	Drava - Ister	Isel - Ister	Tauernbach - Ister
296.586 km	50.094 km	35.885 km	58.252 km

TABLE 3.3.2. The table contains three parts. In the first part, the Hausdorff distances (HD) of the river courses in their full length are presented. Then the partial river courses are compared.

Curve(river)		Maximal HD [km]		Mean HD [km]	
A	B	$\delta_H(A,B)$	$d_H(A,B)$	$\overline{\delta}_H(A,B)$	$\overline{d}_H(A,B)$
D	Ister	296.586		117.455	
Ister	D	205.818	296.586	92.198	108.467
DD	Ister	219.255		84.164	
Ister	DD	205.818	219.255	74.903	80.301
TIDD	Ister	219.255		83.610	
Ister	TIDD	205.818	219.255	74.175	79.679
D1	I1	296.586		145.940	
I1	D1	194.015	296.586	115.984	137.085
Drava	I1	101.197		65.823	
I1	Drava	94.308	101.197	60.811	63.574
TID	I1	101.197		64.181	
I1	TID	94.308	101.197	58.773	61.763
D2	I2	219.255		112.879	
I2	D2	205.818	219.255	99.975	107.675

TABLE 3.3.3. The table contains three parts. In the first part, the matching lengths of the river courses in their full length are presented. Then the partial river courses are compared.

Curve(river)			Matching length $L_m(A, B, d_t)$ [km]		
A	L_A [km]	B	$d_t = 10$	$d_t = 50$	$d_t = 100$
D	2714.887	Ister	46.134 (1.6 %)	327.386 (12.0 %)	911.606 (33.5 %)
Ister	1468.316	D	40.573 (2.7 %)	248.261 (16.9 %)	674.137 (45.9 %)
	A v e r a g e		43.354 (2.0 %)	287.824 (13.7 %)	792.871 (37.9 %)
DD	2021.995	Ister	22.798 (1.1 %)	374.882 (18.5 %)	1390.515 (68.7 %)
Ister	1468.316	DD	20.561 (1.4 %)	301.285 (20.5 %)	1130.088 (76.9 %)
	A v e r a g e		21.679 (1.2 %)	338.083 (19.3 %)	1260.301 (72.2 %)
TIDD	2026.547	Ister	48.427 (2.3 %)	379.434 (18.7 %)	1395.067 (68.8 %)
Ister	1468.316	TIDD	48.975 (3.3 %)	296.829 (20.2 %)	1130.088 (76.9 %)
	A v e r a g e		48.701 (2.7 %)	338.131 (19.3 %)	1262.577 (72.2 %)
D1	1430.501	I1	19.581 (1.3 %)	122.200 (8.5 %)	257.106 (17.9 %)
I1	600.305	D1	10.404 (1.7 %)	57.143 (9.5 %)	144.355 (24.0 %)
	A v e r a g e		14.992 (1.4 %)	89.671 (8.8 %)	200.730 (19.7 %)
Drava	737.609	I1	0.000 (0.0 %)	191.060 (25.9 %)	727.479 (98.6 %)
I1	600.305	Drava	0.000 (0.0 %)	178.483 (29.7 %)	600.305 (100.0 %)
	A v e r a g e		0.000 (0.0 %)	184.771 (27.6 %)	663.892 (99.2 %)
TID	742.161	I1	25.628 (3.4 %)	195.612 (26.3 %)	732.031 (98.6 %)
I1	600.305	TID	28.413 (4.7 %)	174.027 (28.9 %)	600.305 (100.0 %)
	A v e r a g e		27.021 (4.0 %)	184.819 (27.5 %)	666.168 (99.2 %)
D2	1284.386	I2	22.798 (1.7 %)	183.821 (14.3 %)	652.906 (50.8 %)
I2	868.010	D2	20.561 (2.3 %)	122.801 (14.1 %)	529.782 (61.0 %)
	A v e r a g e		21.679 (2.0 %)	153.311 (14.2 %)	591.344 (54.9 %)

4. Discussion on some historical issues

We could finish our work here, just by developing the mathematical model and the numerical algorithm and by showing the correspondence of the Strabo's Ister with the nowadays Tauernbach-Isel-Drava-Danube course. However, since any new result in the mathematical and computational modelling used to bring new insight into the related pure or applied science problem, we are going to do the same for the history which forms our application background.

As we have already stated in the Introduction, many historical claims assuming that in the time of Strabo the upper course of the river Ister corresponds to the nowadays upper course of the Danube river must be revisited. Regarding this fact, there are many interesting open questions arising but two of them are the most interesting for us, (i) where was then located the so-called Hercynian Forest/Hercynia silva/Herkynian Forest (Ἑρκυνίου δρυμος) [19, 30, 35], the seat of Suevi/Suevi/Soebians (Σοήβων) [19, 30, 35] ? And (ii) who were Strabo's **Suevi**?

In the sequel, we will use the terms Suevi and Hercynian Forest because they seem to be the most spread in the English and Latin literature. We also note that in the Slovak literature the terms like Svébi/Suavi are used as well [33]. First of all, at the end of Section (4–6–9) Strabo writes that the Ister source is near the seats of Suevi and the Hercynian Forest:

"ὅπου αἱ τοῦ Ἴστρου πηγαὶ πλησίον Σοήβων καὶ τοῦ Ἑρκυνίου δρυμοῦ".

Since we have shown above where the Ister source is located by means of Strabo's *Geographica*, we can clearly state that Strabo's Suevi near the Ister source have no relation with the Swabia (and the Swabians) in Bavaria, Germany, but we can claim that Strabo is speaking here about a settlement in the south-east Alpine region, around the boundaries of nowadays Carinthia, Tyrol, north-east Italy and Slovenia. With a high probability, this settlement was Slavic in that time and before which can be confirmed by many geographic names around the Strabo's Ister source which are of Slavic origin. The origin of local geographic names in the neighbourhood of Val di Pusteria, in the valleys of upper Drava, Villgraten, Gail and Isel rivers, was studied in [25] where almost 200 names from this local area, including settlements, rivers and creeks, hills, forests or meadows, of the Slavic origin were presented. This study is based on works [23, 24] by Franc Miklošič (Franz Miklosich), one of the most respected philologists of the Habsburg empire in the second half of the 19th century. In two volumes Miklošič presented all the important rules for creating the Slavic geographic

names (Vol. I) and he collected a comprehensive set of 789 bases of Slavic geo-
graphic names (Vol. II) from the whole Habsburg empire. He also gave the most
common rules for changing Slavic names to German (and Hungarian) such as
change of the Slavic "B" to German "F", etc. Now, we present just a few examples
of geographic names of the Slavic origin around the Ister source. Here, and also
in further paragraphs of this section, we give the meaning of these geographic
names together with English also in the Slovak language, because it is the most
familiar to authors and there does not exist any common Slavic language, and
where it has a sense we give it also in the Slovenian due to [25].

First interesting geographic name is Val di Pusteria (Pustertal in German,
Puster Valley in English), which would be in Slovak "Pusté údolie" or "Pustá
dolina" and in Slovenian "Pusta dolina" or "Pustodol", see [25] and [24, point 512]
meaning "Deserted Valley" in English. At the eastern end of the western (lower)
part of the Puster Valley, there is the castle hill, nowadays called Heinfels, be-
low which the river Villgraten(bach) empties to the Drava. By [25] the river
name has the Slavic origin "Velegrad" and there are two other creeks around,
Gradenbach and Gratzbach (mentioned as Gradiz in [25]) with the same Slavic
base "grad", see [24, point 122]. A bit to the west, there is the village Versciaco
(Vierschach in German) with the meaning "Vŕšok" in Slovak and "hillock" in
English, see also [25]. Going further north-east, in the Defereggen Valley there is
the settlement Feistritz, nowadays part of St. Jacob in Defereggen, and also the
creek Feistritz(bach). Feistritz is a German writing of the Slavic name "Bystrica"
in Slovak and "Bistrica" in Slovenian, see [25], [24, point 45]. As a further nice
example we mention Proßegg(-klamm) [25], the village and the gorge on the
Tauernbach creek north of the town Matrei in Osttirol. The name Proßegg has
exact analogy in Prosiek village and gorge in Chočské vrchy, Slovakia, since
"priesek" in Slovak just means "gorge" in English and "klamm" in German. An-
other interesting fact is given by the historical names of Matrei in Osttirol town
and Großvenediger mountain. They were called Windisch Matray als Mauter and
Windisch Taurn, respectively, on the map in Figure 1.3, and it is generally ac-
cepted that "Windisch" meant Slavic in German dialects. Further north, there is
the valley and the long creek Frosnitz(bach) [25], with the source just below the
Großvenediger glaciers, emptying to Tauernbach near Gruben village. The name
Frosnitz is a German writing of the Slavic name "Brusnica" meaning "cranberry"
in English, see [25] and [24, point 33]. Interestingly, another creek with the same
name meaning, the Fruschnitz(bach), is stemming from the western side of Groß-
glockner [25]. But the most astonishing is the name of the longest glacier in
the Eastern Alpine region - Pasterze - in Slovak pronunciation "Pastierce" which
means the place of shepherds or the pastureland. Such form of the place name has
the classical Slavic suffix -ce (-ze in German writing), see [23, Chapter 2, Sect. V,
points 18 and 17]. And why is it so astonishing? Recent results studying peat

samples from the area of the retreating Pasterze glacier indicates grass-like pastureland vegetation (Cyperaceae - *Carex*, *Bidens alba* - shepherd's needles) and human impact on the vegetation during the Subboreal Chronozone (3780–800 BC) a warmer period of the Holocene [16]. In that period only, the name Pasterze with the Slavic pastureland meaning could be given to that place by the inhabitants of this Alpine region - with a very high probability by the Suevi people. They lived there in the Strabo's times and there is no reason to assume that they did not settle there before. We adopt here the hypothesis of the continuity of settlement if no change is recorded in any historical source which is in agreement, e.g., with the Palaeolithic Continuity Theory of Mario Alinei declaring the stability and continuity of Romance-Celtic/Germanic/Slavic ethnic and language geographic distribution in Europe from the Upper Palaeolithic period [1,2]. It is also worth to note that such assumption, assuming stable Slavic sedentary population in this part of Europe already in the Strabo's times, does not exclude any immigration of the same or different ethnic origin and/or local acculturation in the later periods.
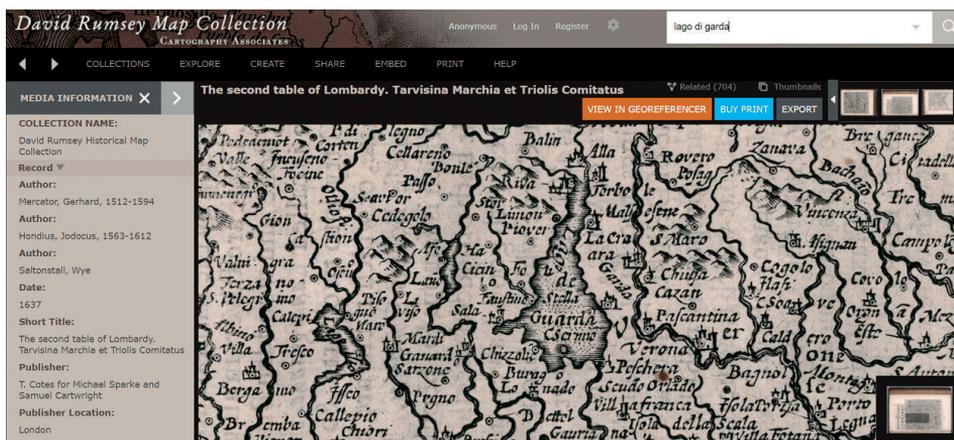
FIGURE 4.1. Detail of the map by Gerhard Mercator from 1639 [22] where in the middle of the map one can see the Sirmione island in Lago di Garda, nowadays connected to the land, where Tiberius could build his military camp in 15 BC during the campaign against Rhaeti.

Next important mentions of the Suevi and Hercynian Forest is in Section (7--1–5) of *Geographica*, where the fights of Tiberius against the Rhaetians around 15 BC are mentioned, see also Cassius Dio [4, Sections (54–22–1)–(54–22–5)] and also [8]. For that military campaigns, Tiberius even built a military camp on the island - Sirmione - of the lake - Lago di Garda, see the map on Figure 4.1 and [8].

By Strabo, the lake is located south of the Ister source and close to the Rhaetian territories (which are around the Tridentine Alps by Cassius Dio [4]), fitting correctly the Lago di Garda and our location of the source of Ister. Then the journey to the Hercynian forest is described by:

"ὥστ᾽ ἀνάγκη τῷ ἐκ τῆς Κελτικῆς ἐπὶ τὸν Ἑρκύνιον δρυμὸν ἰόντι πρῶτον μὲν διαπερᾶσαι τὴν λίμνην, ἔπειτα τὸν Ἴστρον, εἴτ᾽ ἤδη δι᾽ εὐπετεστέρων χωρίων ἐπὶ τὸν δρυμὸν τὰς προβάσεις ποιεῖσθαι δι᾽ ὀροπεδίων."

So, from the (Cisalpine) Keltike, which is for Strabo the north part of nowadays Italy up to the base of the Alps (see Section (5–1–3) of *Geographica*), one has to go along that lake (Lago di Garda) then continue up to and along the course of Ister (taking simply the route along the Adige-Isarco-Rienza-Drava courses) and then continue straightforwardly through more favourable upland planes to end up in the Hercynian Forest. Just looking at any map, e.g., Figure 3.1.1, one can clearly see that this journey must finish in the Carpathian-Alpine basin or better say in the large region between the Eastern Alps on the west and the Carpathian mountains ridge on the east and north. The last sentence of Section (7–1–5) is also very interesting:

"ἔστι δὲ καὶ ἄλλη ὕλη μεγάλη Γαβρῆτα ἐπὶ τάδε τῶν Σοήβων, ἐπέκεινα δ᾽ ὁ Ἑρκύνιος δρυμός: ἔχεται δὲ κἀκεῖνος ὑπ᾽ αὐτῶν."

It is not easy to translate, but in any case, it says that the whole Hercynian Forest, together with another large forest Gabreta (Γαβρῆτα), belongs to and is the seat of the Suevi. In the case that the Gabreta is on the west side of the Hercynian Forest it should correspond to the mountainous forested regions below the main Alpine ridges in the east and north-east directions such as the lower parts of Carinthia, Styria and nowadays Vienna Forest. In the case that the Gabreta is on the east and north sides of the Hercynian Forest, then it should correspond to the Carpathians mountains - Karpaty in the Slovak language. A further indication of the correctness of the location of the Hercynian Forest in between the Alps and the Carpathians is given in Section (7–3–1) of *Geographica*, where Strabo mentions the land of Getians which were assumed to be a Thracians by Hellenes, see (7–3–2):

"οἱ τοίνυν Ἕλληνες τοὺς Γέτας Θρᾷκας ὑπελάμβανον"

and Thracia was generally considered as the land north of Greece up to the Ister (lower Danube). And in (7–3–1) Strabo explicitly says that the land of Getians extends along the southern side of the Istros, nowadays northern Bulgaria, and also on the opposite side, on the mountain slopes of the Hercynian Forest:

"εἶτ᾽ εὐθὺς ἡ τῶν Γετῶν συνάπτει γῆ, κατ᾽ ἀρχὰς μὲν στενή, παρατε-
ταμένη τῷ Ἴστρῳ κατὰ τὸ νότιον μέρος, κατὰ δὲ τοὐναντίον τῇ παρωρείᾳ
τοῦ Ἑρκυνίου δρυμοῦ".

From there, it is obvious that the Getians territory north of the Ister corre-
sponds to the nowadays south part of Romania behind the Carpathian ridges
and it is adjoining the Hercynian Forest. These facts indicate that the Hercynian
Forest, the seat of Suevi, corresponds (at least) to the Carpathian-Alpine basin
including the south-eastern Alps and Carpathian mountains as well.

From the above facts, we can conclude that the Hercynian Forest - the Car-
pathian-Alpine basin in a broad sense - was in the times of Strabo settled by the
Suevi people, the large ethnic group living in this compact area encompassed
by the mountain ranges, as he says in Sections (7–1–3) and (7–1–5) of *Geo-
graphica*. There are many remains of such compact settlement mainly in geo-
graphic names of rivers, mountains, towns and villages in the whole region, see,
e.g. [23, 24, 31, 32]. For interested reader, we are going to touch some of them, con-
centrating our description mainly on the west (Slovenia, Carinthia, Styria), north
(Slovakia) and east (Romania) nodes of an imaginary triangle in the Carpathian-
Alpine basin.

Probably the most spread geographic name of the rivers and settlements are
the names Bystrica/Bystrá (in Slovakia), Bistrica/Bistra (in Slovenia), Bistriţa/
Bistra (in Romania) and Feistritz (in Austria), meaning "quickly flowing"
in Slavic languages, see also [24, Point 45]. In Slovakia, there are five Bystrica and
one Bystrá settlements, for example, towns Banská Bystrica, Považská Bystrica,
etc., two river streams with the name Bystrica, one mountain peak Bystrá
in the West Tatra mountains and there is also saddle Bystré sedlo in the High
Tatra mountains. In Slovenia, there are at least ten towns and villages with the
name Bistrica, e.g., Ilirska Bistrica, Slovenska Bistrica, etc., the river Kamniška
Bistrica and stream Bistra. In Austria, there are at least eight settlements with
the name Feistritz, mainly in Carinthia and Styria, there is a saddle Feistritz-
Sattel, on the border of Styria and Lower Austria, below which is the spring of a
long Styrian river Feistritz. We note that by [24, 25], around 1870–1880 there was
reported Feistritz 15 times in Carinthia and 40 times in Styria. Surprisingly also
in Romania, there are at least seven Bistriţa towns and villages and five rivers
with such name and even more, nine Bistra rivers and three such settlements, dis-
tributed all around the country, in counties Alba, Bacău, Bihor, Bistriţa-Năsăud,
Caraş-Severin, Gorj, Maramureş, Mehedinţi, Mureş, Neamţ, Olt, Sibiu, Suceava,
Vâlcea, and there is also a mountain range called Bistriţa in northern central
Romania. It is worth to note that there are many further such examples of com-
mon geographic names, e.g., "Slatina", with the meaning a "mineralized water"
([24, point 585]) and with the exact same writing at all the places, "Trnava" and
its analogy "Târnava" in Romania, with the meaning of adjective related

to "thorn" ([24, point. 696]). Interestingly, in Romania, there are whole regions with very dense names of villages almost exclusively using the basis of the Slavic words or even more, copying almost exactly the village names used, e.g., in Slovakia. For example, in Caraș-Severin county, near the Valea Cernei (Údolie Čiernej (rieky) in Slovak), are the villages Camena (Kamenná in Slovak), Cozia (Kozia), Dobraia (Dobrá), Dolina (Dolina), Gruni (Grúň), Hora Mare (Veľká Hora), Hora Mică (Malá Hora), Iablanița (Jablonica), Ilova (Ilava), Obița (Obyce), Rusca (Ruská), Rustin (Hruštín), Sadova Nouă (Nová Sadová), Sadova Veche (Stará Sadová), Slatina-Timiș (Timišská Slatina), Studena (Studená), Topla (Teplá), Zbegu (Zbehy), Plugova (Pluhová), Zoina (Zolná), etc., and there exist such examples around all Romania. Another interesting example is the usage of the geographic name "Studena" with the meaning "cold" ([24, point 636]) in practically the same form in Slovakia, Romania, Serbia, Croatia, Slovenia and even in Italy (Studena Alta and Studena Bassa villages in the province of Udine close to Carinthian border) although nowadays in the majority of these languages the word "studena" is not used to express the coldness. After this general overview, we touch further examples of Slavic geographic names which interest us because they appear in Roman and Greek writings from the beginning of the first millennium.
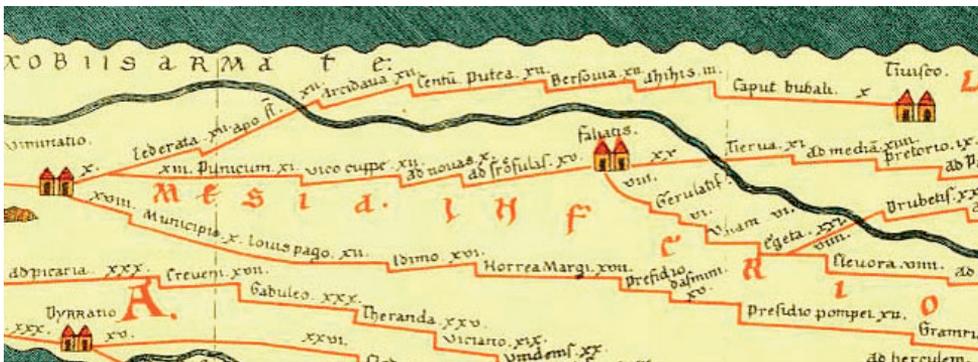
FIGURE 4.2. A detail of Tabula Peutingeriana with Trajan's roads to the province of Dacia and two stations "Bersouia" (on the top road) and "Tierua" (on the second from the top road). Source: Wikipedia.

First, let us take the geographic names derived from the basis of Slavic word "breza" which means "birch" tree ([24, point 29]). In Slovakia, we have towns Brezová pod Bradlom and Brezno and villages Brezovica (twice), Brezovička, Brezov, Rimavské Brezovo and České Brezovo and one river stream Brezovský potok (Brezovka). In Slovenia, there are villages Brezova, Brezova Reber pri Dvoru, Brezno (twice), Breza and Brezovo. In Romania there are several forms
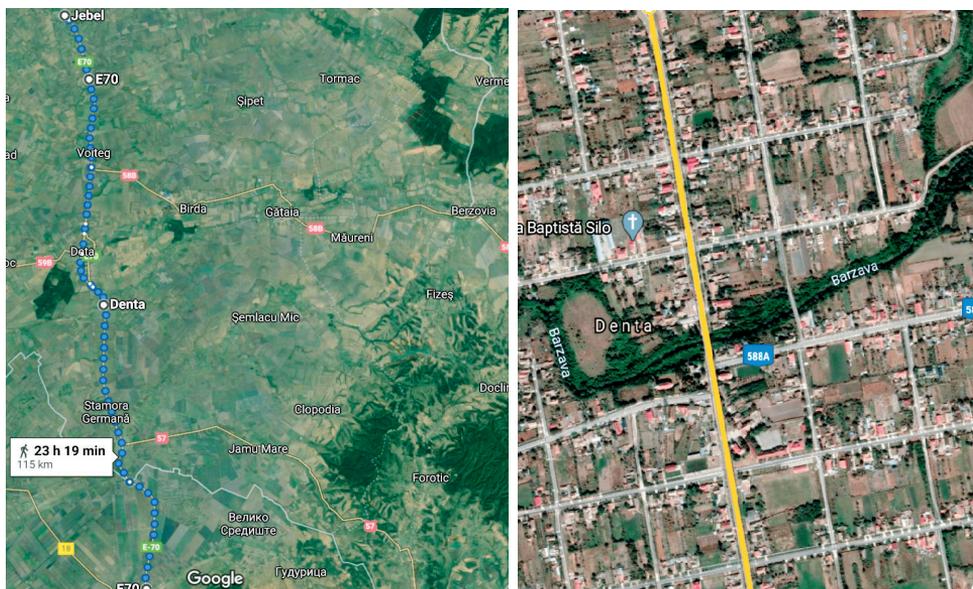
FIGURE 4.3. A detail of the estimated Trajan's road to "Tiuisco" (Timișoara) on Tabula Peutingeriana (left image) with four stops indicated, "Centu Putea" (E70 mark on the bottom), "Bersouia" (Denta), "Azizis" (E70 mark on the top) and "Caput bubali" (Jebel). Station "Bersovia" is on the road crossing with the Bârzava river in the village Denta (right image). The village Berzovia which is also located on the river Bârzava is more on the east and the road going through it would not fulfil the distances indicated on the Tabula Peutingeriana. That's why we think that the station called Berzovia was at the place where the road crossed the river with the same name.

of this name, there are seven towns or villages and three rivers with the name Breaza, two villages and two river streams with the name Breazova, one of which is a tributary of the river Bârzava, which is another form of the same name. The river Bârzava, flowing through historical regions of Banat (Romania) and Vojvodina (Serbia), has further tributaries Bârzăvița and Berzovița, there is the village Berzovia on the river Bârzava and very near is another village Brezon. It used to be claimed that the village Berzovia on the river Bârzava is noted on the Tabula Peutingeriana [39] as the station "Bersouia", see, e.g., Pavol Jozef Šafárik seminal work [36]. Tabula Peutingeriana shows the Roman road system in the first centuries AD and it should be last revised in the 4th or early 5th century. The "Bersouia" is one of the stations on the most north Trajan's road to the province of Dacia which crosses the Ister (lower Danube) near the Banatska Palanka and continues in the direction to "Tiuisco" (Timișoara), see Fig. 4.2.

In fact, the "Bersouia" is mentioned also in Trajan's work *Dacica*, from around 100 AD, as "inde Berzobim, deinde Aizi processimus", meaning going from "Bersouia" to "Azizis", see Fig. 4.2. Taking into account the Tabula Peutingeriana distances given in Roman miles we claim that "Bersouia" or "Bersobis" is not directly the Berzovia village but it is (with high probability) the crossing point of that Trajan's road with the Bârzava river at a place near (or in) the Denta village, see Fig. 4.3. Such claim, however, does not change anything on the interesting fact that the widely spread in Carpathian basin geographic name of the Slavic origin is mentioned on the map from the first centuries AD and in the work of emperor Trajan from around 100 AD.

There is a mountainous river Belá, with the meaning "white", stemming just below Kriváň, one of the highest peaks of the High Tatra mountains and thus of the whole Carpathians. Belá got her name probably by the white smooth rocks in its river-basin. The similar is true for the northern Italian, the Carnic Alpine river Fella, with the Slovenian name Bela. Fella has its spring close to the Friuli-Carinthia-Slovenian border and concerning the current name it went through the standard change of the Slavic "B" to the German "F" recorded now in the Italian official name, see [23] and [24, point 12]. This twin Belá-Bela is another clear example of the same geographic name given to mountain rivers by the common Slavic population living in the Alps and Carpathians. Moreover, very near the Fella (Bela) spring, there is the Carinthian town Villach - Beljak in Slovenian. And reading carefully the Tabula Peutingeriana one can find it also there under the name "Beliandro", see Fig. 4.4. We see again the Slavic name of the settlement recorded on the map describing the Roman road system at the beginning of the first millennium AD. We can simply check our claim that Villach (Beljak) corresponds to the "Beliandro" by computing distance from Ptuj ("Petauione" on Tabula Peutingeriana) through Celje ("Celeia") to Villach ("Beliandro"). The whole distance is 137 Roman miles. In order to check the relation of the length of the Roman mile and the Google map distances in kilometres in this area of Tabula Peutingeriana we do it for two clearly given towns and known distances both in kilometres and in the Roman miles. From Ptuj to Celje we have 58,5 km on Google maps and 36 miles on Tabula Peutingeriana, which gives approximate correspondence 1.62 km for 1 Roman mile. Now, 137×1.62 km $= 222, 24$ km and its there, see Fig. 4.5, the distance from Ptuj to Villach is 220 km on Google maps which perfectly approximate the Tabula Peutingeriana distance 137 Roman miles. Of course, one can check the Villach-"Beliandro" correspondence also in different ways, e.g., by estimating forward distances from "Beliandro" to other stations on the Roman road, but that we left for a reader.

After finding the Belá-Bela twin in the High Tatras and the Carnic Alps one can try to find a corresponding name also in Romania. In the Caraş-Severin county, there is a river Belareka meaning "Biela rieka" in Slovak and "White river"

FIGURE 4.4. A detail of Tabula Peutingeriana with the road from "Petauione" through "Celeia" to "Beliandro" and farther away. Also stations "Tergeste" and "fonte timaui" can be seen on this map detail in the left bottom part. Source: Wikipedia.
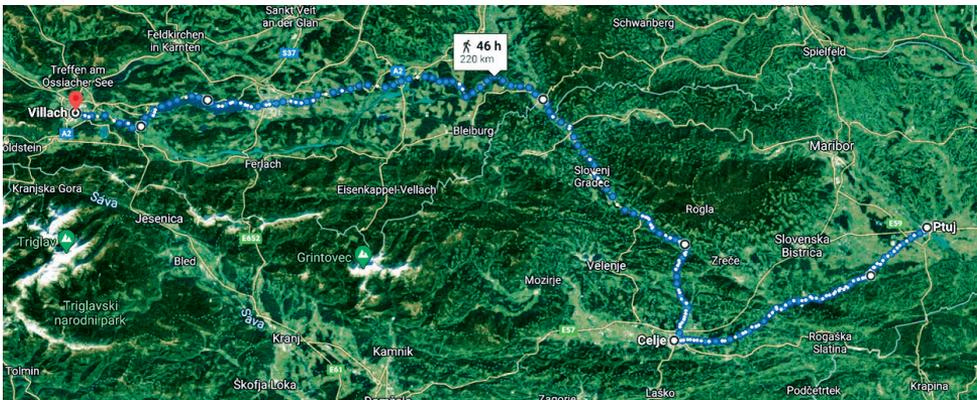
FIGURE 4.5. Distance from Ptuj to Villach through Celje is 220 km by the Google maps while by the Tabula Peutingeriana it should be 137 Roman miles which is approximately 222 km. Such accurate correspondence of distances shows that nowadays Villach (Beljak) corresponds to the "Beliandro" on the Tabula Peutingeriana.

in English but written with both words together. It is again a mountainous river which joins the river Cerna ("Čierna" in Slovak, [24, point 71], "Black" in English) near the spa town Baile Herculane. The Cerna river then flows to the Ister and their confluence in nowadays Orșova got the name "Tierua" on Tabula Peutingeriana, see Fig. 4.2. It represents the place of crossing the Ister (lower Danube) by the second Trajan's road to the Dacia province. Interestingly, the place name was also recorded by Klaudios Ptolemaios in Greek as "Δίερνα", see [9, 36].

Since in the Greek alphabet there is no direct representation of the letter "č", Ptolemaios replaced it by "Δ" due to a pronunciation similarity of Čierna-Δίερνα. This record had to be written between AD 85–165, during the Ptolemaios life and after approximately AD 100 when Trajan's roads to Dacia were constructed. We also know by Pavol Jozef Šafárik [36] that there exists inscription on marble in Mehadia, former Roman camp on Belareka, from AD 157, "Valerius Felix miles coh. IV. stationis TSIERNEN", where "Tsiernen" represents a very accurate transcript of the river Cerna (Čierna) name to the Latin.

All these Slavic names writings are dated a few decades later than Strabo's *Geographica* was written. Since by Strabo we know that Suevi inhabited that region in the beginning of the first millennium AD, we clearly see the correspondence between the Strabo's Suevi in the Hercynian forest and the Slavs or Slavic settlements in the Carpathian-Alpine basin. We may also claim that the name Suevi (with the Latin "v") gave birth to two nowadays ethnic names of Slavic nations from the former Hercynian Forest, Slovenes and Slovaks and the name form Soebi (with the Greek "b") to the third one, the Serbians which are called "Srbi" in Slovak (in pronunciation quite similar to "Soebi"). And in fact, in the Middle Ages Latin sources up to the end of the 18th century, the Slovak people were called Slavi, Sclavi or gentis Slavae (in pronunciation again very similar to Suevi or Suavi, when considering the term "Suavia" used in [33], page 33), see, e.g., "Privilegium pro Slavis solnensis" by Ľudovít I. Veľký (Louis the Great from Anjou) from 1381, where Slovaks are called Sclavi [17,18] or "Historia gentis Slavae" by Juraj (Georgius) Papánek from 1780 [27].

Almost finally we mention "Tergeste" geographic name used by Strabo in *Geographica* and appearing also on Tabula Peutingeriana, see Fig. 4.4 bottom left. The name "Tergeste" has correspondences in Romania, such as Târgovişte in Dâmboviţa County, Muntenia historical region of Romania, which was also the capital of Wallachia between the early 15th century and the 16th century. And the exact correspondence exists also in Slovakia, the village Trhovište. The common meaning of all these places is the "marketplace" and the name is derived from the Slavic equivalent "trh", "trg" or "targ", see [24, point 694]. There are several further examples such as Târgu Jiu, Târgu Neamţ, Târgu Mureș, Târgu Frumos or Târgu Secuiesc in Romania. By the current name similarity, the "Tergeste" used to be related to nowadays Trieste but with some probability, it may also correspond to the city of Monfalcone which is called Tržič in Slovenian with the marketplace meaning [24, point 694]. Concerning Tabula Peutingeriana, we can find several further names of clear Slavic origin in Slavonia and Bosnia regions, see Figure 4.6. There is the station "Vrbate" on the crossing of the Roman road with the nowadays river Vrbas, with the Slavic base of the name, "vŕba" (in Slovak) meaning the "willow" tree, see [24, pont 746]. Then there is the "pont Vlcae" station close to nowadays town Vukovar on the Vuka river, with the Slavic

base "vlk" (in Slovak) or "vuk" (in Croatian and Serbian) meaning the "wolf", see [24, point 733]. A further example is the station "Drinum fl" at the place of nowadays Drina river, with the Slavic base "drieň" (in Slovak) meaning the "bunchberry", see [24, point 87].

The last but not least we mention the karst river Timavo which re-emerges near Monfalcone and empties into the Adriatic. Strabo calls it Timavus in Section (5–1–8) of *Geographica* and he says it has seven sources by which it re-emerges after 130 stadia in the underground. The Slovenian name of the river is Timava, with the standard Slavic suffix -ava, see [23, Chapter 2, Section V, point 37]. And the meaning can be "Tmavá" (in Slovak) translated as "dark". This name meaning nicely corresponds to the re-emergence of Timava from the "dark underground". This Slavic name of the river appears also on the Tabula Peutingeriana as "fonte timaui", see Fig. 4.4 bottom left, and it again represents a clear example of the Slavic presence in the south-eastern Alpine region at the beginning of the first millennium AD.

FIGURE 4.6. A detail of Tabula Peutingeriana with stations "Vrbate" (left), "pont Vlcae" (middle) and "Drinum fl." (right) in Bosnia and Slavonia. Source: Wikipedia.

5. Conclusions

All in all, we have shown the correspondence of the *S*trabo's Suevi of the Hercynian Forest with the Slavs in the Carpathian-Alpine basin. In other words, we confirmed the presence of the compact Slavic settlement in this region already at the beginning of the first millennium AD. Interestingly, such a conclusion comes from our mathematical results on the transformation of the Strabo's river Ister to the current map of the world. To perform the transformation we developed a new method for the map registration combining the locally optimal affine transformations with the interpolation/extrapolation by solving numerically the Laplace equation with zero Neumann boundary conditions.

The new method keeps the optimality of the local transformations and smoothly interpolate/extrapolate the transformation parameters to the whole computational domain. The method was applied to Strabo's river Ister reconstruction and yields interesting historical conclusions.

The conclusions are in accordance with the opinions and claims of historians and linguists such as Pavol Jozef Šafárik [36], Ľudovít Štúr [7, 37, 38], Oleg Trubačev [40] or Mario Alinei [1, 2] which all declared the ancientness of the Slavs on the middle Danube, i.e., in the Carpathian-Alpine basin. It supports in some aspects also the narratives of Primary chronicle by the Saint Nestor the Chronicler [26] seeing the middle Danube even as the homeland of all the Slavs, which is however for us too strong and hardly acceptable assumption. More likely, we are in accordance with several aspects of the Paleolithic Continuity Paradigm by Mario Alinei [1, 2] declaring stability of the European population and its ethnic distribution from the very ancient (Upper Paleolithic or Mesolithic) times. Such claims are also in accordance with the results of [29] where the authors concluded that the large majority of extant human mitochondrial DNA (mtDNA) lineages entered Europe in several waves during the Late Upper Palaeolithic, mainly around the Last Glacial Maximum, 20000 years ago. After that, the bearers of the mtDNA - sedentary population - adopted in general to a new way of life and the Neolithic demic-diffusion, see, e.g. [28], and further immigration waves seem to contribute only by about 10–20 % of mtDNA lineages. Also, recent work [5] of Slovak and Hungarian experts in archaeology and archaeogenetics has shown a large majority of mtDNA haplogroups in medieval (9th–12th century AD) population around Nitra (Western Slovakia) belonging to the Late Upper Palaeolithic period of migration to Europe and similarity of medieval "Slovak" population with nowadays inhabitants of the Carpathian-Alpine basin such as Croats and Romanians.

We hope that results of this paper can help geographers and cartographers with the historical map registration and also to historians to localize the "Suavia" at the beginning of the first millennium AD [33] or to explain how it was possible that Slavs appeared so fast in the Carpathian basin and north of the Istros (Lower Danube) at the beginning of the second half of the first millennium AD [6, 14]. In fact, the Slavs did not appear, they were there "since time immemorial".

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