Deterministic self-similar models of complex networks based on very symmetric graphs

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Abstract

Using very symmetric graphs we generalize several deterministic self-similar models of complex networks and we calculate the main network parameters of our generalization. More specifically, we calculate the order, size and the degree distribution, and we give an upper bound for the diameter and a lower bound for the clustering coefficient. These results yield conditions under which the network is a self-similar and scale-free small world network. We remark that all these conditions are posed on a small base graph which is used in the construction. As a consequence, we can construct complex networks having prescribed properties. We demonstrate this fact on the clustering coefficient. We propose eight new infinite classes of complex networks. One of these new classes is so rich that it is parametrized by three independent parameters.

1 Introduction

In last years, many real-life networks from very different areas were studied, see e.g. [13], and it was observed that, typically, these networks have some common properties. They have small average degree, small distances between the vertices and big clustering. More precisely, if n is the order (the number of vertices) of the network then:

(A1) The number of edges is in $O(n \ln n)$.

(A2) The diameter is in $O(\ln n)$.

(A3) For the clustering coefficient $C(G)$ we have $C(G) \geq c$ for some positive constant c.

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These three properties define *small world networks* as proposed by Watts and Strogatz in [15]. Later, Barabási and Albert observed that many complex networks are scale-free. More precisely:

(A4) The proportion of vertices of degree at least k is approximately equal to $k^{1-\gamma}$, where γ typically satisfies $2 \leq \gamma \leq 3$.

Next property of complex-networks is self-similarity, see [14].

There appeared many random models of networks satisfying (A1)–(A4) and also several deterministic ones. The deterministic models, whose advantage is that their properties can often be computed analytically, have usually some common features. We define here a general construction, such that the networks introduced in [7, 12, 22, 23, 24], and with a slight modification also those of [17, 18, 19, 20], are special cases of our construction. However, there are some deterministic constructions, such as the hierarchic models of [3, 4] and the model based on edge valuations [21], which we do not generalize.

After introducing our model, we calculate the number of vertices and edges of the network, we find the degree distribution and we calculate γ from (A4). We also find a good upper bound for the diameter and a lower bound for the clustering coefficient, so that (A2) and (A3) can be checked easily. Then we demonstrate our results on three previously invented models. Our choice of these models was such that they are as different as possible. As expected, all our results agree for these networks.

Finally, based on our general construction, we introduce eight new deterministic models of complex networks. All our constructions yield infinite classes of networks, and one of them, namely Construction 6, can be parametrized by three independent parameters. Non self-repetitive version of another one, namely Construction 5, generalizes both the Farey network and the Apollonian network constructions, see [12, 1]. In the process of modelling these constructions we focussed our attention to the clustering coefficient. Three of these new constructions have clustering coefficient close to 1 and three have this coefficient close to $\frac{1}{2}$. The first three are based on the complete graph K_k and the second three are based on the complete tripartite graph $K_{k,k,k}$. The last two constructions have clustering coefficient 0. One is based on the complete bipartite graph $K_{k,k}$ and one on the graph of a prism. These last two constructions do not satisfy (A3), but we include them here as some of the previous models of self-similar networks also have clustering coefficient 0 (to demonstrate this fact see Constructions 1 and 2 below). We point out that in all these new models all the parameters mentioned above are obtained by simple substitution of relevant constants to the derived formulae.

2 Three previously invented models

We recall here three constructions of self-similar networks. The first construction appeared in Zhang and Comellas [16], the second in Comellas, Zhang and Chen [8] and the third in

Comellas, Fertin and Raspaud [6]. However, we define these constructions in slightly different words than in [16, 8, 6]. The reason for this is that we like to point out their common features, which are generalized in the next section. For illustration of these constructions see Figure 1.

Construction 1. Let H be a graph with one special edge $\mathcal{U}$, endvertices of which are connected by j internally-vertex-disjoint paths of length 3. Thus, if $j = 1$ then H is a square, while if $j > 1$ then H consists of j squares sharing the edge $\mathcal{U}$. Denote by $\mathcal{T}$ the set of edges of H that contain exactly one vertex of $\mathcal{U}$. Then $\mathcal{T}$ has $2j$ edges and we call them active edges. The construction is following:

- If $t = 0$, $G_j(0)$ has two vertices connected by an active edge.
- If $t > 0$, $G_j(t)$ is obtained from $G_j(t-1)$ by identifying every active edge of $G_j(t-1)$ with the edge $\mathcal{U}$ of a copy of H. Hence, if $G_j(t-1)$ contains $q(t-1)$ active edges, we glue to $G_j(t)$ exactly $q(t-1)$ new copies of H. In $G_j(t)$, the active edges are exactly the edges of $\mathcal{T}$'s in just attached $q(t-1)$ copies of H.

Construction 2. Let H be the graph of a cube, that is, H has 8 vertices all of which have degree 3. Denote by $\mathcal{U}$ one square of H. Let $\mathcal{T}$ be the set of squares of H which share exactly one edge with $\mathcal{U}$. Then $\mathcal{T}$ has four squares and we call them active squares. The construction is following:

- If $t = 0$, $G(0)$ consists of one active square.
- If $t > 0$, $G(t)$ is obtained from $G(t-1)$ by identifying every active square of $G(t-1)$ with the square $\mathcal{U}$ of a copy of H, so that the edges of $\mathcal{U}$ are identified with the edges of the active square of $G(t-1)$. Hence, if $G(t-1)$ has $q(t-1)$ active squares, we glue to $G(t-1)$ exactly $q(t-1)$ copies of H. In $G(t)$ the active squares are exactly the squares of $\mathcal{T}$'s in just attached $q(t-1)$ copies of H.

Construction 3. Let H be the complete graph on $j+1$ vertices, K_{j+1}, and let $\mathcal{U}$ be one of its induced subgraphs on j vertices. Then $\mathcal{U}$ is a complete graph on j vertices K_j. Denote by $\mathcal{T}$ the set of all induced j-vertex subgraphs of H, including $\mathcal{U}$ itself. The complete graphs of $\mathcal{T}$ are active. The construction is following:

- If $t = 0$, $G_j(0)$ consists of an active complete graph on j vertices.
- If $t > 0$, $G_j(t)$ is obtained from $G_j(t-1)$ by identifying every active K_j with $\mathcal{U}$ of a copy of H. Hence, if $G_j(t-1)$ has $q(t-1)$ active K_j's, we glue to $G_j(t-1)$ exactly $q(t-1)$ copies of H. In $G_j(t)$, active copies of K_j are exactly the graphs of $\mathcal{T}$'s in just attached copies of H.

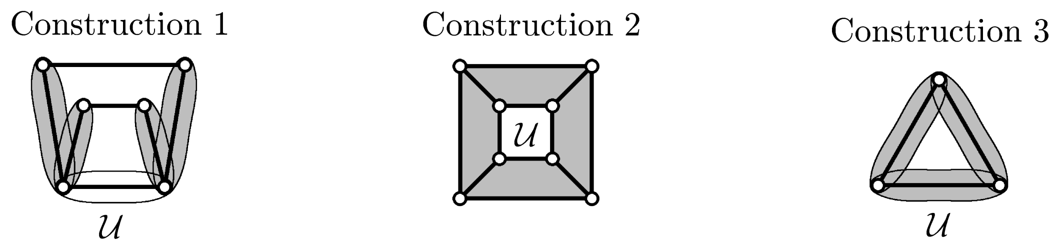

Figure 1: Graphs H for each of Constructions 1–3, where $j = 2$ in Constructions 1 and 3. The graph $\mathcal{U}$ is an edge in Constructions 1 and 3 and this edge is circled. In Construction 2 the graph $\mathcal{U}$ is a 4-cycle. Copies of $\mathcal{U}$ belonging to $\mathcal{T}$ are shaded.

In our notation, $G(0)$ is isomorphic to $\mathcal{U}$ and $G(1)$ is isomorphic to H in Constructions 1–3. In Construction 3, if some copy of K_j is active once, then it is active forever. Therefore we call Construction 3 *self-repetitive*, while Constructions 1 and 2 are *non self-repetitive*. In [6], Construction 3 is defined in a slightly different way. It says that in t-th iteration we find all the copies of K_j in $G_j(t-1)$ and to all vertices of every such K_j we connect one new vertex. But it is easy to see that our definition is equivalent with this one.

3 Generalized deterministic model of a self-similar network

In this section we will unify the constructions introduced in the previous section. For this, we recall here some notions used in graph theory. Let G be a graph with vertex set $V(G)$. An *automorphism* of G is a bijective mapping $\varphi : V(G) \to V(G)$, such that if uv is an edge in G, $u, v \in V(G)$, then also $\varphi(u)\varphi(v)$ is an edge of G. The set of all automorphisms of G is denoted by $\mathrm{Aut}\,(G)$. If there exists $\varphi \in \mathrm{Aut}\,(G)$ such that $\varphi(x) = y$, then we say that x and y belong to the same *orbit* of $\mathrm{Aut}\,(G)$. If there is just one orbit in $\mathrm{Aut}\,(G)$, then G is *vertex-transitive*. More about algebraic graph theory can be found in [9].

Though Constructions 1–3 are rather different, they have similar features. In every case, the vertices of $\mathcal{U}$ belong to the same orbit of the group of automorphisms of H, $\mathrm{Aut}\,(H)$, and so $\mathcal{U}$ is a vertex-transitive graph. Also, the vertices of H which are not in $\mathcal{U}$ belong to one orbit of $\mathrm{Aut}\,(H)$. Thus, there are at most two orbits in $\mathrm{Aut}\,(H)$ and consequently, vertices of H have at most two distinct degrees. Nevertheless in each of these three constructions, vertices of the network $G(t)$ which appear in the i-th iteration do not belong to one orbit of $\mathrm{Aut}\,(G(t))$ although these vertices all have the same degree. To satisfy these degree conditions, every vertex of $\mathcal{U}$ must be incident with the same number of active copies of $\mathcal{U}$, and analogous property must hold for the vertices of H which are not in $\mathcal{U}$. Moreover, since the construction is deterministic, if we attach H to $G(t-1)$ via $\mathcal{U}$, the resulting structure with active copies of $\mathcal{U}$ must be independent of the way of attaching. That is, every automorphism of $\mathcal{U}$ must be extendable to such an automorphism of H, which maps the elements of $\mathcal{T}$ to themselves. Now we summarize all these requirements and we introduce parameters q, r and s used throughout the rest of this paper.

Definition 1. *Let H be a graph with a subgraph $\mathcal{U}$, such that there is at least one vertex of H which is not in $\mathcal{U}$. Further, let $\mathcal{T}$ be a set of (not necessarily all) copies of $\mathcal{U}$ in H. The graphs of $\mathcal{T}$ are the active copies of $\mathcal{U}$. We call the triple $(H,\mathcal{U},\mathcal{T})$ an S-*structure*, if there are parameters $q, r \geq 2$ and $s \geq 1$ and the following are true:*

- *All the vertices of $\mathcal{U}$ belong to one orbit of H.*
- *All the vertices of H which are not in $\mathcal{U}$ belong to one orbit of H.*
- *Every automorphism of $\mathcal{U}$ is extendable to such an automorphism of H which maps the graphs of $\mathcal{T}$ to themselves.*
- *The set $\mathcal{T}$ contains exactly q copies of $\mathcal{U}$.*
- *Every vertex of $\mathcal{U}$ is incident with r active copies of $\mathcal{U}$.*
- *Every vertex of H which is not in $\mathcal{U}$ is incident with s active copies of $\mathcal{U}$.*

Of course, Constructions 1–3 satisfy the requirements above, which can be easily verified on Figure 1. Now we are in a position to introduce a construction which generalizes Constructions 1–3.

Construction 4. Let H be a graph with a subgraph $\mathcal{U}$, and with a set $\mathcal{T}$ of copies of $\mathcal{U}$, such that $(H,\mathcal{U},\mathcal{T})$ is an S-structure. The construction is following:

- If $t = 0$, $G(0)$ consists of an active copy of $\mathcal{U}$.
- If $t > 0$, $G(t)$ is obtained from $G(t-1)$ by identifying every active copy of $\mathcal{U}$ with $\mathcal{U}$ in a copy of H. Hence, if $G(t-1)$ has $q(t-1)$ active copies of $\mathcal{U}$, we glue to $G(t-1)$ exactly $q(t-1)$ new copies of H. In $G(t)$, the active copies of $\mathcal{U}$ are exactly the graphs of $\mathcal{T}$'s in the $q(t-1)$ attached H's.

In Figure 2 we have a schematic description of Construction 4. Obviously, $G(t)$ is a self-similar network which generalizes Constructions 1–3. However, it also generalizes five constructions of deterministic networks introduced in [7, 12, 22, 23, 24], and the networks studied in [17, 18, 19, 20] differ only in the graph $G(0)$ which is not isomorphic with $\mathcal{U}$. However, if $G(0) = \mathcal{U}$, we have a nice recursive description of Construction 4.

Recursive modular construction. The graph $G(t)$ can also be defined as follows:

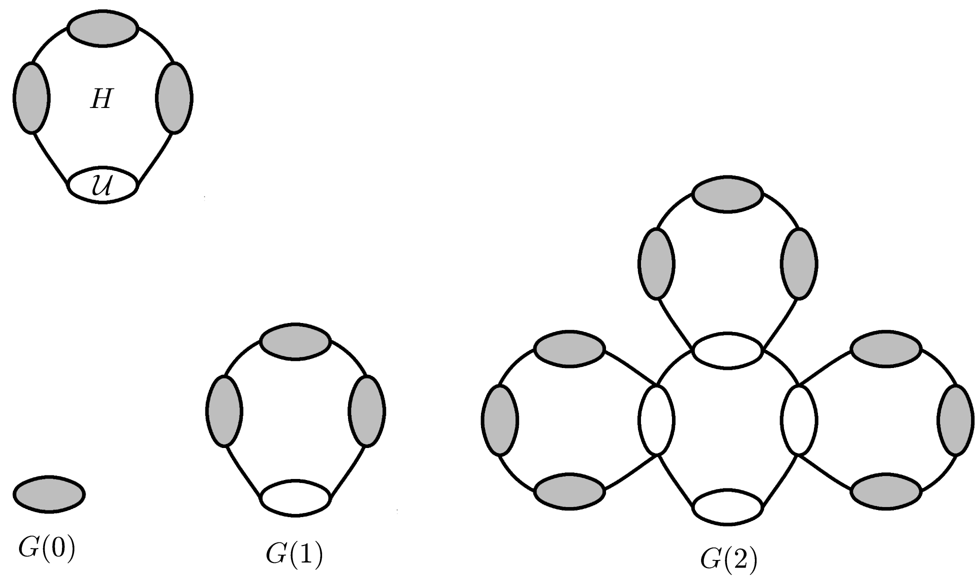

Figure 2: Schematic description of Construction 4. Above is H with $\mathcal{U}$ and $\mathcal{T}$, where $\mathcal{T}$ consists of the shaded copies of $\mathcal{U}$. Below are $G(0)$, $G(1)$ and $G(2)$, and in all these graphs the shaded regions represent the active copies of $\mathcal{U}$.

- If $t = 0$, $G(0)$ consists of an active copy of $\mathcal{U}$. We denote this copy by $\overline{\mathcal{U}}$.
- If $t = 1$, $G(1)$ is obtained from $G(0)$ by identifying $\overline{\mathcal{U}}$ with $\mathcal{U}$ in a copy of H.
- If $t \geq 2$, $G(t)$ is obtained from $G(1)$ and q copies of $G(t-1)$ by identifying every active copy of $\mathcal{U}$ in $G(1)$ with $\overline{\mathcal{U}}$ in a copy of $G(t-1)$.

We remark that the recursion step has also a more general version:

- If $t \geq 2$, then for every i, $1 \leq i < t$, $G(t)$ is obtained from $G(i)$ and several copies of $G(t-i)$ by identifying every active copy of $\mathcal{U}$ in $G(i)$ with $\overline{\mathcal{U}}$ in a copy of $G(t-i)$.

4 Properties of the generalized construction

Though Construction 4 is rather general, we are able to calculate the main network parameters of $G(t)$. Particularly, we can state conditions under which (A1)–(A4) are satisfied. We prove here the following theorem by a set of claims:

Theorem 1. *Let $(H, \mathcal{U}, \mathcal{T})$ be an S-system and let q, r and s be as in Definition 1. Moreover, let c be the clustering coefficient of such a vertex of H which is not in $\mathcal{U}$. Then Construction 4 yields a model of a self-similar network with properties (A1)–(A4) if the parameters satisfy $q \geq 2$, $r \geq 2$, $s \geq 1$, $1 \leq \frac{\ln(q)}{\ln(r)} \leq 2$ and $c > 0$.*

Observe that the parameters q, r, s and c can be verified by analyzing H, together with $\mathcal{U}$ and $\mathcal{T}$. That is, we do not need to consider $G(t)$. This allows to propose networks with specified properties as will be demonstrated in Section 6.

Now we study the order, size, degree distribution, diameter and the clustering coefficient. All the notation introduced below will be used through the rest of the paper.

Order of $G(t)$. Let $q(t)$ be the total number of active copies of $\mathcal{U}$ in $G(t)$. Then $q(t) = q \cdot q(t-1) = q^2 \cdot q(t-2) = \cdots = q^t \cdot q(0) = q^t$ as $G(0)$ consists of a unique active copy of $\mathcal{U}$. Denote by $N(t)$ the number of vertices of $G(t)$. Further, denote by N_0 the number of vertices of $\mathcal{U}$ and denote by N_1 the number of vertices of H. Then $N(0) = N_0$ and $N(1) = N_1 = N_0 + (N_1 - N_0)$. For $t > 1$ we have $N(t) = N(t-1) + q(t-1)(N_1 - N_0)$ since to $G(t-1)$ we attach $q(t-1)$ copies of H, each with $(N_1 - N_0)$ new vertics. Since $q(t-1) = q^{t-1}$ and $q \geq 2$, we get $N(t) = N(t-1) + q^{t-1}(N_1 - N_0) = \cdots = N_0 + q^0(N_1 - N_0) + q^1(N_1 - N_0) + \cdots + q^{t-1}(N_1 - N_0) = N_0 + \frac{q^t - 1}{q-1}(N_1 - N_0)$. Hence, we have shown:

Claim 1. *The order $N(t)$ satisfies*

$$N(t) - N(t-1) = q^{t-1}(N_1 - N_0) \qquad \text{and} \qquad N(t) = N_0 + \frac{q^t - 1}{q - 1}(N_1 - N_0).$$

Observe that for arbitrary graph G and two of its vertices, say x and y, if both x and y belong to a common orbit of $\mathrm{Aut}\,(G)$, then they have the same degree. This implies that all the vertices of $\mathcal{U}$ have the same degree and also all vertices of H which are not in $\mathcal{U}$ have the same degree. Now we find the size of $G(t)$.

Size of $G(t)$. Let d_0 be the degree of a vertex in $\mathcal{U}$. Further, let d_1 be the degree of a vertex of H which is in $\mathcal{U}$, and let d_e be the degree of a vertex of H which is not in $\mathcal{U}$. Denote by $M(t)$ the number of edges of $G(t)$. Further, denote by F the graph obtained from H by removing the edges of $\mathcal{U}$. Then F has N_0 vertices of degree $(d_1 - d_0)$ and $(N_1 - N_0)$ vertices of degree d_e. Thus, F has $\frac{1}{2}[(d_1 - d_0)N_0 + d_e(N_1 - N_0)]$ edges. If we attach to a network a copy of H through $\mathcal{U}$, then the new edges are exactly the edges of F. Therefore $M(t) = M(t-1) + q(t-1)\frac{1}{2}[(d_1 - d_0)N_0 + d_e(N_1 - N_0)]$. Since $M(0) = \frac{1}{2}d_0 N_0$, we have:

Claim 2. *The size $M(t)$ satisfies*

$$M(t) = \frac{1}{2}d_0 N_0 + \frac{q^t - 1}{q - 1}\frac{1}{2}\Big[(d_1 - d_0)N_0 + d_e(N_1 - N_0)\Big].$$

Hence $M(t) \in O(N(t)) \subseteq O(N(t)\ln[N(t)])$, which means that $G(t)$ allways satisfies (A1).

Degree distribution. The degrees are in Table 1, where in the $G(t)$ case we have $1 \le i \le t-1$.

Graph	its degrees	numbers of vertices
$G(0)$	d_0	N_0
$G(1)$	$d_0 + (d_1 - d_0)$	N_0
	d_e	$N_1 - N_0$
$G(2)$	$d_0 + (d_1 - d_0) + r(d_1 - d_0)$	N_0
	$d_e + s(d_1 - d_0)$	$N_1 - N_0$
	d_e	$q(N_1 - N_0)$
$G(3)$	$d_0 + (d_1 - d_0) + r(d_1 - d_0) + r^2(d_1 - d_0)$	N_0
	$d_e + s(d_1 - d_0) + rs(d_1 - d_0)$	$N_1 - N_0$
	$d_e + s(d_1 - d_0)$	$q(N_1 - N_0)$
	d_e	$q^2(N_1 - N_0)$
$G(t)$	$d_0 + \frac{r^t-1}{r-1}(d_1 - d_0)$	N_0
	$d_e + \frac{r^{t-i}-1}{r-1}s(d_1 - d_0)$	$q^{i-1}(N_1 - N_0)$
	d_e	$q^{t-1}(N_1 - N_0)$

Table 1: Degrees of vertices of the network.

Let $\Delta(t)$ be the maximum degree of $G(t)$. From Table 1 it follows that $\Delta(t) = \max\{d_0 + \frac{r^t-1}{r-1}(d_1-d_0), d_e + \frac{r^{t-1}-1}{r-1}s(d_1-d_0)\}$. The degrees $d_e + \frac{r^{t-i}-1}{r-1}s(d_1-d_0)$ grow in a very regular way for $1 \le i \le t-1$. However, even if $\Delta(t) = d_0 + \frac{r^t-1}{r-1}(d_1-d_0)$, then $\Delta(t)$ cannot "jump" much out of this regularity. Since $s \ge 1$, we have $d_e \ge d_0$. Hence, $d_0 + \frac{r^t-1}{r-1}(d_1-d_0) \le d_e + \frac{r^t-1}{r-1}s(d_1-d_0)$ and consequently $\Delta(t) \le d_e + \frac{r^t-1}{r-1}s(d_1-d_0)$.

Denote by $N(k,t)$ the number of vertices of degree k in $G(t)$. We find γ such that

$$\sum_{k' \ge k} N(k',t)/N(t) \sim k^{1-\gamma}. \tag{1}$$

From Table 1, there are $q^{i-1}(N_1-N_0)$ vertices of degree $d_e + \frac{r^{t-i}-1}{r-1}s(d_1-d_0)$, $1 \le i \le t-1$. Out of this regular sequence there are $q^{t-1}(N_1-N_0)$ vertices of degree d_e, but these vertices are not important for (1) as their degree is constant. There are also N_0 vertices of degree $d_0 + \frac{r^t-1}{r-1}(d_1-d_0)$, but they are as well not important for (1) as their number is constant. Let $k = d_e + \frac{r^{t-i}-1}{r-1}s(d_1-d_0)$ for some i, where $1 \le i \le t-1$. Then

$$\sum_{k' \ge k} N(k',t) \sim \Big[q^{i-1} + q^{i-2} + \cdots + 1\Big](N_1 - N_0) = \frac{q^i - 1}{q-1}(N_1 - N_0).$$

Substituting to (1) yields

$$\left(d_e + \frac{r^{t-i}-1}{r-1}s(d_1-d_0)\right)^{1-\gamma} \sim \left[\frac{q^i-1}{q-1}(N_1-N_0)\right] \Big/ \left[\frac{q^t-1}{q-1}(N_1-N_0)+N_0\right].$$

For t large we get $(s(d_1-d_0)r^{t-i-1})^{1-\gamma} \sim q^{i-t}$ and so

$$1-\gamma \sim \frac{\ln(q^{i-t})}{\ln\big(s(d_1-d_0)r^{t-i-1}\big)} \sim \frac{(i-t)\ln(q)}{\ln\big(s(d_1-d_0)\big)-\ln(r)+(t-i)\ln(r)} \sim -\frac{\ln(q)}{\ln(r)}.$$

We get:

Claim 3. *For γ we have*

$$\gamma \sim 1+\frac{\ln(q)}{\ln(r)}.$$

Thus, if $r \geq 2$ and $1 \leq \frac{\ln(q)}{\ln(r)} \leq 2$, then $G(t)$ satisfies (A4).

We remark that all the networks introduced in Constructions 5–11 below satisfy the inequality $1 \leq \frac{\ln(q)}{\ln(r)} \leq 2$. In the last construction we demonstrate that Construction 4 can yield networks with arbirtarily large γ, so the last construction does not satisfy $\frac{\ln(q)}{\ln(r)} \leq 2$.

Diameter. Denote by $D(H)$ the diameter of H and denote by $D(t)$ the diameter of $G(t)$. Since $G(1)$ is a graph isomorphic to H, we have $D(1)=D(H)$. In H, the eccentricity of every vertex of $\mathcal{U}$ is at most $D(H)$, and so $D(t) \leq D(t-1)+2D(H) \leq \cdots \leq (2t-1)D(H)$. We get:

Claim 4. *For the diameter $D(t)$ we have*

$$D(t) \leq (2t-1)D(H).$$

If $q \geq 2$ then $N(t)$ grows exponentially and hence $D(t) \in O(\ln(N(T)))$, which means that $G(t)$ satisfies (A2).

We remark that in some cases the bound $D(t) \leq (2t-1)D(H)$ is tight, see Construction 5 below.

Clustering coefficient. Clustering coefficient of a vertex equals the number of edges in its (open) neighbourhood, divided by $\binom{d}{2}$, where d is its degree. Hence, clustering coefficient is the proportion of the number of existing edges to the number of all possible edges in the neighbourhood of a vertex. Clustering coefficient of a network is the average of clustering coefficients taken over all the vertices of the network.

Let $C(t)$ be the clustering coefficient of $G(t)$. Recall that c is the clustering coefficient of a vertex of H which is not in $\mathcal{U}$. There are $N(t)-N(t-1)=q^{t-1}(N_1-N_0)$ vertices in $G(t)$

with clustering coefficient equal to c, while all the other vertices have clustering coefficient at least 0. By Claim 1, we have $N(t) = \frac{q^t-1}{q-1}(N_1 - N_0) + N_0$, and so

$$C(t) \geq \frac{c[N(t) - N(t-1)]}{N(t)} = \frac{c\left[q^{t-1}(N_1 - N_0)\right]}{\frac{q^t-1}{q-1}(N_1 - N_0) + N_0}. \tag{2}$$

Since for $q \to \infty$ the limit of the right-hand side of (2) is $\frac{c(q-1)}{q}$, we have shown the following:

Claim 5. *For the clustering coefficient $C(t)$ we have*

$$C(t) \geq c\Big(1 - \frac{1}{q}\Big).$$

Hence if $q \geq 2$ and $c > 0$, then $C(t) > 0$, and consequently $G(t)$ satisfies (A3).

5 Revisiting the three previously invented models

We demonstrate the results of the previous section on Constructions 1–3. That is, using the parameters q, r and s defined in Definition 1 and d_0, d_1, d_e, N_0 and N_1 defined in the previous section, we calculate the order, size, the degrees, and the bounds from Claims 4 and 5 and we compare these results with those found in [16, 8, 6].

In Construction 1 we have $q = 2j$, $r = j$, $s = 1$, $d_0 = 1$, $d_1 = j+1$, $d_e = 2$, $N_0 = 2$ and $N_1 = 2+2j$. Thus $N(t) = N_0 + \frac{q^t-1}{q-1}(N_1 - N_0) = \frac{(2j)^{t+1}+2j-2}{2j-1}$ and $M(t) = \frac{1}{2}d_0N_0 + \frac{q^t-1}{q-1}\frac{1}{2}[(d_1-d_0)N_0 + d_e(N_1 - N_0)] = \frac{3j(2j)^t-j-1}{2j-1}$. There are $N_0 = 2$ vertices of degree $d_0 + \frac{r^t-1}{r-1}(d_1 - d_0) = \frac{j^{t+1}-1}{j-1}$; $N(i) - N(i-1) = q^{i-1}(N_1 - N_0) = (2j)^i$ vertices of degree $d_e + \frac{r^{t-i}-1}{r-1}s(d_1 - d_0) = 1 + \frac{j^{t-i+1}-1}{j-1}$, where $1 \leq i \leq t-1$; and $N(t) - N(t-1) = (2j)^t$ vertices of degree $d_e = 1 + \frac{j-1}{j-1}$. Next, $\gamma = 1 + \frac{\ln(q)}{\ln(r)} = 1 + \frac{\ln(2j)}{\ln(j)}$, and all these parameters agree with [16]. We have $D(t) \leq (2t+1)D(H) = 4t+2$, but in fact $D(t) = 2t+1$. Since H does not contain triangles, we have $c = 0$ and consequently we obtain the trivial inequality $C(t) \geq 0$. In fact, the clustering coefficient of $G(t)$ is 0.

In Construction 2 we have $q = 4$, $r = 2$, $s = 2$, $d_0 = 2$, $d_1 = 3$, $d_e = 3$, $N_0 = 4$ and $N_1 = 8$. Thus $N(t) = N_0 + \frac{q^t-1}{q-1}(N_1 - N_0) = \frac{4^{t+1}+8}{3}$ and $M(t) = \frac{1}{2}d_0N_0 + \frac{q^t-1}{q-1}\frac{1}{2}[(d_1-d_0)N_0 + d_e(N_1 - N_0)] = \frac{2\cdot 4^{t+1}+4}{3}$. There are $N_0 = 4$ vertices of degree $d_0 + \frac{r^t-1}{r-1}(d_1 - d_0) = 2^t + 1$; $N(i) - N(i-1) = q^{i-1}(N_1 - N_0) = 4^i$ vertices of degree $d_e + \frac{r^{t-i}-1}{r-1}s(d_1 - d_0) = 2^{t-i+1} + 1$, where $1 \leq i \leq t-1$; and $N(t) - N(t-1) = 4^t$ vertices of degree $d_e = 3$. Next, $\gamma = 1 + \frac{\ln(q)}{\ln(r)} = 1 + \frac{\ln(4)}{\ln(2)} = 3$, and all these parameters agree with [8]. We have $D(t) \leq (2t+1)D(H) = 6t+3$, but in fact $D(t) = 2t+1$. Since H does not contain triangles, we have $c = 0$ implying the trivial inequality $C(t) \geq 0$. In fact $C(t) = 0$.

In Construction 3 we have $q = j+1$, $r = j$, $s = j$, $d_0 = j-1$, $d_1 = j$, $d_e = j$, $N_0 = j$ and $N_1 = j+1$. Thus $N(t) = N_0 + \frac{q^t-1}{q-1}(N_1 - N_0) = j + \frac{(j+1)^t-1}{j}$ and $M(t) = \frac{1}{2}d_0N_0 + \frac{q^t-1}{q-1}\frac{1}{2}[(d_1-d_0)N_0 + d_e(N_1-N_0)] = \frac{j(j-1)}{2} + (j+1)^t - 1$. There are $N_0 = j$ vertices of degree $d_0 + \frac{r^t-1}{r-1}(d_1-d_0) = j-1+\frac{j^t-1}{j-1}$; $N(i)-N(i-1) = q^{i-1}(N_1-N_0) = (j+1)^{i-1}$ vertices of degree $d_e + \frac{r^{t-i}-1}{r-1}s(d_1-d_0) = j + \frac{j^{t-i}-1}{j-1}j$, where $1 \le i \le t-1$; and $N(t)-N(t-1) = (j+1)^{t-1}$ vertices of degree $d_e = j$. Next, $\gamma = 1 + \frac{\ln(q)}{\ln(r)} = 1 + \frac{\ln(j+1)}{\ln(j)}$, and all these parameters agree with [6]. We have $D(t) \le (2t+1)D(H) = 2t+1$, but in fact $D(t) \sim 2t/j + 2$. For the clustering coefficient we have $C(t) \ge 1 - \frac{1}{j+1}$, but in fact $C(t) \ge \frac{3j-2}{3j-1} = 1 - \frac{1}{3j-1}$.

6 New deterministic models of self-similar networks

Now we introduce several new deterministic models of self-similar networks. All they are special cases of Construction 4.

We start by choosing complete graph on k vertices for H, since then the clustering coefficient of H is $c = 1$, and hence we can expect the largest clustering coefficient in the network. In our calculations we omit the size and the degrees, but it is easy to find them by substituting the relevant constants to the formulae derived above.

Construction 5. Let H be the complete graph on k vertices, $k \ge 3$, and let $\mathcal{T}$ be the set of all j-vertex induced subgraphs of H, $j \ge 2$. Further, let $\mathcal{U}$ be one of the graphs of $\mathcal{T}$. Obviously, every automorphism φ of $\mathcal{U}$ is extendable to such an automorphism of H, which preserves $\mathcal{T}$ (it suffices to choose the extension of φ outside $\mathcal{U}$ to be the identity). Since the other properties of Definition 1 are trivially satisfied, $(H,\mathcal{U},\mathcal{T})$ is an S-structure. Thus, we can apply Construction 4 on $(H,\mathcal{U},\mathcal{T})$, and we denote the resulting construction as Construction 5.

Observe that Construction 5 is a straightforward generalization of Construction 3, which is obtained when $k = j+1$. We have $q = \binom{k}{j}$, $r = s = \binom{k-1}{j-1}$, $d_0 = j-1$, $d_1 = d_e = k-1$, $N_0 = j$ and $N_1 = k$. Thus $N(t) = N_0 + \frac{q^t-1}{q-1}(N_1-N_0) = j + \frac{\binom{k}{j}^t-1}{\binom{k}{j}-1}$. Next, $\gamma = 1 + \frac{\ln(q)}{\ln(r)} = 1 + \frac{\ln\binom{k}{j}}{\ln\binom{k-1}{j-1}}$, and so $2 < \gamma < 3$. We have $D(t) \le (2t+1)D(H) = 2t+1$. However, if $k \ge 2j$ then $D(t) \ge D(t-1)+2$, and so $D(t) = 2t+1$ in this case. Finally, $C(t) \ge c(1-\frac{1}{q}) = 1 - 1/\binom{k}{j}$, as the clustering coefficient c of every vertex in a complete graph is 1.

When we change Construction 5 from self-repetitive to non self-repetitive, we obtain very similar values of the above mentioned parameters. This modification generalizes the Farey network construction ($k = 3$ and $j = 2$) and the Apollonian network construction ($k = 4$ and $j = 3$), see [12, 1].

We can modify Construction 5 by forbiding some configurations, to obtain the next construction.

Construction 6. Let H be the complete graph on k vertices, $k \geq 3$, and let ℓ divides k. Let the vertex set of H be decomposed into k/ℓ sets of size ℓ and let j satisfies $2 \leq j \leq k/\ell$. Denote by $\mathcal{T}$ the set of all induced j-vertex subgraphs of H, which do not contain a pair of vertices from a common set of the decomposition. Further, denote by $\mathcal{U}$ one of the graphs of $\mathcal{T}$. Obviously, every automorphism of $\mathcal{U}$ is extendable to such an automorphism of H, which preserves $\mathcal{T}$ (it suffices to respect the decomposition of the vertex set of H). Since the other properties of Definition 1 are trivially satisfied, $(H,\mathcal{U},\mathcal{T})$ is an S-structure. Thus, apply Construction 4 on $(H,\mathcal{U},\mathcal{T})$ and denote the resulting construction as Construction 6.

Let a, b and c be three integer parameters, $a, b \geq 2$ and $c \geq 0$. Set $\ell = a$, $j = b$ and $k/\ell = b + c$. Then $k = a(b + c)$ and for different values of a, b and c we get different instances of Construction 6. Hence, this construction can be parametrized by three independent parameters.

Observe that if $\ell = 1$, Construction 6 reduces to Construction 5. We have $q = \binom{k/\ell}{j}\ell^j$, $r = s = \binom{k/\ell-1}{j-1}\ell^{j-1}$, $d_0 = j - 1$, $d_1 = d_e = k - 1$, $N_0 = j$ and $N_1 = k$. Thus $N(t) = N_0 + \frac{q^t-1}{q-1}(N_1 - N_0) = j + \frac{\left[\binom{k/\ell}{j}\ell^j\right]^t-1}{\binom{k/\ell}{j}\ell^j-1}(k - j)$. Next, $\gamma = 1 + \frac{\ln(q)}{\ln(r)} = 1 + \frac{\ln\binom{k/\ell}{j}+j\ln(\ell)}{\ln\binom{k/\ell-1}{j-1}+(j-1)\ln(\ell)}$, and so $2 < \gamma \leq 3$. We have $D(t) \leq 2t + 1$ and $C(t) \geq 1 - [\binom{k/\ell}{j}\ell^j]^{-1}$.

Probably the most interesting cases ocure when k is small:

- For $k = 4$ and $\ell = j = 2$ we get $q = 4$, $r = s = 2$, $d_0 = 1$, $d_1 = d_e = 3$, $N_0 = 2$ and $N_1 = 4$. So $N(t) = 2 + \frac{4^t-1}{3}2$, $\gamma = 3$, $D(t) \leq 2t - 1$ and $C(t) \geq 3/4$.
- For $k = 6$, $\ell = 3$ and $j = 2$ we get $q = 9$, $r = s = 3$, $d_0 = 1$, $d_1 = d_e = 5$, $N_0 = 2$ and $N_1 = 6$. So $N(t) = 2 + \frac{9^t-1}{2}$, $\gamma = 3$, $D(t) \leq 2t - 1$ and $C(t) \geq 8/9$.
- For $k = 6$, $\ell = 2$ and $j = 3$ we get $q = 8$, $r = s = 4$, $d_0 = 2$, $d_1 = d_e = 5$, $N_0 = 3$ and $N_1 = 6$. So $N(t) = 3 + \frac{8^t-1}{7}3$, $\gamma = 1 + \frac{3}{2} = 2.5$, $D(t) \leq 2t - 1$ and $C(t) \geq 7/8$.

Construction 7. Let $k = 2^\ell - 1$, $\ell \geq 3$. We denote the vertices of K_k, $V(K_k)$, by 0-1 vectors of length ℓ, avoiding the vector having all coordinates 0. That is, $V(K_k) = \mathbb{Z}_2^\ell \setminus (0, 0, \ldots, 0)$. Let $\mathcal{T}$ be the set of triples of vectors x, y, z of $V(K_k)$, such that when summing every coordinate in $\mathbb{Z}_2$, we get $(0, 0, \ldots, 0)$. Then $\mathcal{T}$ forms a projective Steiner triple system, see e.g. [5]. Every pair of vertices of K_k is in a unique triple of $\mathcal{T}$ and it is known that $\mathcal{T}$ is doubly-transitive, which means that if a mapping φ maps an ordered pair of vertices to any other (but fixed) pair of ordered vertices, then φ can be extended to an automorphism of $\mathcal{T}$. Let $H = K_k$ and let $\mathcal{U}$ be one of the triples of $\mathcal{T}$. Since every mapping of $\mathcal{U}$ to itself is determined uniquely by the images of two vertices of $\mathcal{U}$, doubly-transitivity of $\mathcal{T}$ implies that every automorphism of $\mathcal{U}$ is extendable to an automorphism of H which preserves the triples of $\mathcal{T}$. Since the other properties of Definition 1 are trivially satisfied, $(H,\mathcal{U},\mathcal{T})$ is an S-structure. Thus, apply Construction 4 on $(H,\mathcal{U},\mathcal{T})$ and denote the resulting construction as Construction 7.

We have $q = \binom{k}{2}/3 = \frac{k(k-1)}{6}$, $r = s = \frac{k-1}{2}$, $d_0 = 2$, $d_1 = d_e = k-1$, $N_0 = 3$ and $N_1 = k$. Thus $N(t) = N_0 + \frac{q^t-1}{q-1}(N_1 - N_0) = 3 + \frac{\left(\frac{k(k-1)}{6}\right)^t - 1}{\frac{k(k-1)}{6} - 1}(k-3)$. Next, $\gamma = 1 + \frac{\ln(q)}{\ln(r)} = 1 + \ln(\frac{k(k-1)}{6}) / \ln(\frac{k-1}{2})$, and so $2 < \gamma \leq 3$. We have $D(t) \leq 2t+1$ and $C(t) \geq 1 - \left[\frac{k(k-1)}{6}\right]^{-1}$.

Our second choice for H is the complete tripartite graph $K_{k,k,k}$. Its vertex set consists of three disjoint subsets of order k, and a pair of vertices is connected by an edge if and only if the vertices belong to distinct subsets. The clustering coefficient of H is $c = k^2/\binom{2k}{2} = \frac{1}{2} + \frac{1}{4k-2}$, and so we can expect that the networks will have clustering coefficient close to $\frac{1}{2}$. Observe that the graph of a regular octahedron is $K_{2,2,2}$.

Construction 8. Let H be the complete tripartite graph on $3k$ vertices, $k \geq 2$, and let $\mathcal{U}$ be one of the triangles of H. Let $\mathcal{T}$ be the set of all triangles of H, including $\mathcal{U}$. Obviously, every automorphism of $\mathcal{U}$ is extendable to an automorphism of H which preserves the triples of $\mathcal{T}$. Since the other properties of Definition 1 are trivially satisfied, $(H,\mathcal{U},\mathcal{T})$ is an S-structure. Thus, apply Construction 4 on $(H,\mathcal{U},\mathcal{T})$ and denote the resulting construction as Construction 8.

We have $q = k^3$, $r = s = k^2$, $d_0 = 2$, $d_1 = d_e = 2k$, $N_0 = 3$ and $N_1 = 3k$. Thus $N(t) = N_0 + \frac{q^t-1}{q-1}(N_1 - N_0) = 3 + \frac{k^{3t}-1}{k^3-1}3(k-1)$. Next, $\gamma = 1 + \frac{\ln(q)}{\ln(r)} = 1 + \frac{\ln k^3}{\ln k^2} = 2.5$, and so $2 < \gamma < 3$. Since $D(H) = 2$, we have $D(t) \leq (2t+1)D(H) = 4t+2$. Finally, $C(t) \geq \frac{k}{2k-1}(1 - \frac{1}{q}) = \frac{1}{2} + \frac{k^2-2}{4k^3-2k^2}$.

A map is an embedding of a graph into a surface (compact 2-manifold) such that when we cut the surface along the embedded edges, the pieces of the surface (faces) will be homeomorphic to open discs. If all the faces are bounded by exactly 3 edges, then the map is a triangulation of the surface. If a map has the property that for every two triples (v_1, e_1, f_1) and (v_2, e_2, f_2), where e_i is an edge incident with the vertex v_i and the face f_i, $1 \leq i \leq 2$, there exists an automorphism φ of the map mapping v_1 to v_2, e_1 to e_2 and f_1 to f_2, then the map is called regular, see e.g. [11]. (An automorphism of a map is an automorphism of the underlying graph which maps faces to faces.)

Construction 9. In [10] it is proved that for every $k \geq 1$ there is a unique regular triangulation of $H = K_{k,k,k}$ in an orientable surface. Let $\mathcal{T}$ be the set of all facial triangles of such a map and $k \geq 2$. Denote by $\mathcal{U}$ one triangle of $\mathcal{T}$. Since every automorphism of $\mathcal{U}$ is determined by the image of one of its vertices and an incident edge, every automorphism of $\mathcal{U}$ is extendable to such an automorphism of $K_{k,k,k}$ which preserves the elements of $\mathcal{T}$ (the faces). Since the other properties of Definition 1 are trivially satisfied, $(H,\mathcal{U},\mathcal{T})$ is an S-structure. Thus, apply Construction 4 on $(H,\mathcal{U},\mathcal{T})$ and denote the resulting construction as Construction 9.

Since $K_{k,k,k}$ has $3k^2$ edges, we have $q = 2(3k^2)/3 = 2k^2$, $r = s = 2k$, $d_0 = 2$, $d_1 = d_e = 2k$, $N_0 = 3$ and $N_1 = 3k$. Thus $N(t) = N_0 + \frac{q^t-1}{q-1}(N_1 - N_0) = 3 + \frac{(2k^2)^t-1}{2k^2-1}3(k-1)$. Next,

$\gamma = 1 + \frac{\ln(q)}{\ln(r)} = 3 - \frac{\ln 2k^2}{\ln 2k}$, and so $2 < \gamma < 3$. We have $D(t) \leq 4t+2$ and $C(t) \geq \frac{k}{2k-1}(1-\frac{1}{q}) = \frac{1}{2} + \frac{k-1}{4k^2-2k}$.

Let $k = 2$. Then the map used in Construction 9 is the regular octahedron. In this case, non self-repetitive version of Construction 9 differs from the construction present in [22] only by one active triangle (the one oposite to $\mathcal{U}$).

Construction 10. The map used in Construction 9 is face two-colourable, see [10]. That is, we can colour its faces by two colours, say black and white, so that every white triangle shares edges only with black triangles and vice-versa. Denote by $\mathcal{T}$ the set of white triangles of this map and denote by $\mathcal{U}$ one white triangle of $\mathcal{T}$. Further, denote $H = K_{k,k,k}$, where $k \geq 2$. Since every automorphism of the map which maps $\mathcal{U}$ to itself maps white triangles to white triangles and black triangles to black ones, $(H, \mathcal{U}, \mathcal{T})$ is an S-structure. Thus, apply Construction 4 on $(H, \mathcal{U}, \mathcal{T})$ and denote the resulting construction as Construction 10.

We have $q = k^2$, $r = s = k$, $d_0 = 2$, $d_1 = d_e = 2k$, $N_0 = 3$ and $N_1 = 3k$. Thus $N(t) = N_0 + \frac{q^t-1}{q-1}(N_1 - N_0) = 3 + \frac{k^{2t}-1}{k^2-1}3(k-1)$. Next, $\gamma = 1 + \frac{\ln(q)}{\ln(r)} = 3$, and so $2 < \gamma \leq 3$. We have $D(t) \leq 4t+2$ and $C(t) \geq \frac{k}{2k-1}(1-\frac{1}{q}) = \frac{1}{2} + \frac{k-2}{4k^2-2k}$.

We remark that the triangles $\mathcal{T}$ in Construction 10 form the cyclic Latin square, see [10]. Observe that if we choose $\mathcal{U}$ to be not white, but a black triangle, then the parameters will remain completely unchanged, but the resulting construction will be non self-repetitive.

Our last two constructions are based on graphs with clustering coefficients 0. We consider such graphs here as also the networks from Constructions 1 and 2 have the clustering coefficient 0.

Construction 11. Let H be the complete bipartite graph $K_{k,k}$ on $2k$ vertices, $k \geq 2$. The vertex set of $K_{k,k}$ consists of two disjoint subsets of order k, and a pair of vertices is connected by an edge if and only if the vertices belong to distinct subsets. Let j satisfy $1 \leq j < k$, and let $\mathcal{U}$ be an induced subgraph of H having j vertices in each of the two subsets defining the bipartition of H. Then $\mathcal{U}$ is isomorphic to $K_{j,j}$. Let $\mathcal{T}$ contain all subgraphs of H isomorphic to $K_{j,j}$. Obviously, every automorphism of $\mathcal{U}$ is extendable to such an automorphism of H, which maps the graphs of $\mathcal{T}$ to themselves (it suffices to respect the bipartition of H). Since the other properties of Definition 1 are trivially satisfied, $(H, \mathcal{U}, \mathcal{T})$ is an S-structure. Thus, apply Construction 4 on $(H, \mathcal{U}, \mathcal{T})$ and denote the resulting construction as Construction 11.

We have $q = \binom{k}{j}^2$, $r = s = \binom{k}{j}\binom{k-1}{j-1}$, $d_0 = j$, $d_1 = d_e = k$, $N_0 = 2j$ and $N_1 = 2k$. Thus $N(t) = N_0 + \frac{q^t-1}{q-1}(N_1 - N_0) = 2j + \frac{\binom{k}{j}^{2t}-1}{\binom{k}{j}^2-1}2(k-j)$. Next, $\gamma = 1 + \frac{\ln(q)}{\ln(r)} = 1 + \frac{2\ln\binom{k}{j}}{\ln\binom{k}{j} + \ln\binom{k-1}{j-1}}$, and so $2 < \gamma < 3$. Since $D(H) = 2$, we have $D(t) \leq (2t+1)D(H) = 4t+2$. Finally, $C(t) \geq 0(1-\frac{1}{q}) = 0$, and it is easy to see that in fact $C(t) = 0$.

Construction 12. Let H be the graph of a k-sided prism. Then H has $2k$ vertices and $3k$

edges and for every edge e there is an automorphism of H which interchanges the endvertices of e. Denote one edge of H by $\mathcal{U}$ and denote the set of all edges of H by $\mathcal{T}$. Then $(H,\mathcal{U},\mathcal{T})$ is an S-structure. Thus, apply Construction 4 on $(H,\mathcal{U},\mathcal{T})$ and denote the resulting construction as Construction 12.

We have $q = 3k$, $r = s = 3$, $d_0 = 1$, $d_1 = d_e = 3$, $N_0 = 2$ and $N_1 = 2k$. Thus $N(t) = N_0 + \frac{q^t-1}{q-1}(N_1 - N_0) = 2 + \frac{(3k)^t-1}{3k-1}2(k-1)$. Next, $\gamma = 1 + \frac{\ln(q)}{\ln(r)} = 1 + \frac{\ln 3k}{\ln 3}$, and so $\gamma \to \infty$ as $k \to \infty$. Since $D(H) = \lfloor \frac{k}{2} \rfloor$, we have $D(t) \leq (2t+1)\lfloor \frac{k}{2} \rfloor$. Finally, if $k \geq 4$ then $c = 0$ and consequently also $C(t) = 0$.

7 Conclusion

In this paper we invented a construction which generalizes some of the previous models of deterministic self-similar networks, and we found several invarians of this construction. Our results allow designing models of complex networks with specific parameters, which we partially demonstrated on the clustering coefficient in Constructions 5–12.

As regards further generalizations, our calculations of $N(t)$, $M(t)$, degree distribution, γ, $D(t)$ and $C(t)$ in Construction 4 can be provided analogously if $G(0)$ is different from an active copy of $\mathcal{U}$, although the formulae will be a little bit more complicated. However, to find the correlation coefficient or the strength distribution, we need more information about H. Hence, these invariants should be calculated separately for every single construction.

By our opinion, the biggest disadvantage of Construction 4 (and all the constructions it generalizes) is that it produces networks with too many symmetries (automorphisms). So a further generalization of Construction 4, such that the resulting self-similar network can be rigid (that is, can have just the trivial automorphism), is a challenge for the future research.

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References

[1] J. Andrade, H. Herrmann, R. Andrade, L. da Silva, Apollonian networks: Simultaneously scale-free, small world, Euclidean, space filling, and with matching graphs Phys. Rev. Lett. 94 (2005), 018702.

[2] A. Barabási, R. Albert, Emergence of scaling in random networks, Science 286 (1999), 509–512.

[3] A. Barabási, E. Ravasz, T. Vicsek, Deterministic scale-free networks, Phys. A 299 (2001), 559–564.

[4] L. Barière, F. Comellas, C. Dafó, M. Fiol, Deterministic hierarchical networks, (manuscript).

[5] C. J. Colbourn, A. Rosa, Triple Systems, Oxford University Press, Oxford, 1999.

[6] F. Comellas, G. Fertin, A. Raspaud, Recursive graphs with small-world scale-free properties, Phys. Rev. E 69 (2004), 037104.

[7] F. Comellas, A. Miralles, Modeling complex networks with self-similar outerplanar unclustered graphs, Phys. A 388 (2009), 2227–2233.

[8] F. Comellas, Z. Zhang, L. Chen, Self-similar non-clustered planar graphs as models for complex networks, J. Phys. A-Math. Theor. 42 (2009), 045103.

[9] C. Godsil, G. Royle, Algebraic Graph Theory, Springer-Verlag, New York, 2001.

[10] M. J. Grannell, T. S. Griggs, M. Knor, J. Širáň, Triangulations of orientable surfaces by complete tripartite graphs, Discrete Math. 306 (2006), 600–606.

[11] J. L. Gross, T. W. Tucker, Topological Graph Theory, John Wiley, New York, 1987.

[12] A. Miralles, F. Comellas, L. Chen, Z. Zhang, Planar unclustered graphs to model technological and biological networks, Phys. A 389 (2010), 1955–1964.

[13] M. Newman, The structure and function of complex networks, Siam Rev. 45 (2003), 167–256.

[14] C. Song, S. Havlin, H. Makse, Self-similarity of complex networks, Nature 433 (2005), 392–395.

[15] D. Watts, S. Strogatz, Collective dynamics of 'small world' networks, Nature 393 (1998), 440–442.

[16] Z. Zhang, F. Comellas, Farey graphs as models for complex networks, Theor. Comput. Sci. 412 (2011), 865–875.

[17] Z. Zhang, L. Rong, F. Comellas, Evolving small-world networks with geographical attachment preference, J. Phys. A, Math. Gen. 39 (2006), 3253.

[18] Z. Zhang, L. Rong, C. Guo, A deterministic small-world network created by edge iterations, Physica A 363 (2006), 567–572.

[19] Z. Zhang, L. Rong, S. Zhou, A general geometric growth model for pseudofractal scale-free web, Phys. A 377 (2007), 329–339.

[20] Z. Zhang, S. Zhou, L. Chen, Evolving pseudofractal networks, Eur. Phys. J. B 58 (2007), 337–344.

[21] Z. Zhang, S. Zhou, L. Chen, J. Guan, L. Fang, Y. Zhang, Recursive weighted treelike networks, Eur. Phys. J. B 59 (2007), 99–107.

[22] Z. Zhang, S. Zhou, L. Fang, J.Guan, Y. Zhang, Maximal planar scale-free Sierpinski networks with small-world effect and power-law strength-degree correlation, Europhys. Lett. 79 (2007), 38007.

[23] Z. Zhang, S. Zhou, T. Zou, L. Chen, J. Guan, Incompatibility networks as models of scale-free small-world graphs Eur. Phys. J. B 60 (2007), 259–264.

[24] Z. Zhang, S. Zhou, W. Xie, L. Chen, Y. Lin, J. Guan, Standard random walks and trapping on the Koch network with scale-free behavior and small-world effect Phys. Rev. E 79 (2009), 061113.