EFFICIENT OPEN DOMINATION IN DIGRAPHS

Ľudovít Niepel[a)] and Martin Knor[b)]

[a)] Kuwait University, Faculty of Science, Department of Mathematics & Computer Science, P.O. box 5969 Safat 13060, Kuwait,
E-mail: NIEPEL@sci.kuniv.edu.kw;

[b)] Slovak University of Technology, Faculty of Civil Engineering, Department of Mathematics, Radlinského 11, 813 68 Bratislava, Slovakia,
E-mail: knor@math.sk.

Abstract. Let G be a digraph. A set $S \subseteq V(G)$ is called an efficient total dominating set if the set of open out-neighborhoods $N^-(v) \in S$ is a partition of $V(G)$. We say that G is efficiently open-dominated if both G and its reverse digraph G^- have an efficient total dominating set. Some properties of efficiently open dominated digraphs are presented. Special attention is given to tournaments and directed tori.

1. Introduction

Dominating sets are widely applied in the design and efficient use of computer networks. They can be used to decide the placement of limited resources so that every node has access to the resource through neighboring node (see for example [5] and [6]). The most efficient solution is one that avoids duplication of access to the resources. This more restricted version of minimum dominating set is called an independent perfect or efficient dominating set. Efficient dominating sets were introduced independently in [1] and [9]. There are many results about efficient dominating sets in different classes of graphs like trees, Cartesian products and Cayley graphs. They can be found for instance in [3], [10], [11]. In a similar way one can define efficient total dominating sets, where each vertex in a graph is dominated by exactly one vertex from total dominating set. These sets are also called perfect dominating sets [7] or efficient open dominating sets [4]. All efficient dominating sets in graph G have the same cardinality that is equal to $\gamma(G)$, the domination number of G. The same property of unique cardinality holds for efficient

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total dominating sets in a graph [4]. In a similar way, efficient dominating sets are defined for directed graphs in [2]. Existence and properties of efficient dominating sets in digraphs are studied for example in [8], [12], [14]. Unfortunately, in a digraph, efficient dominating sets of different cardinalities can exist. Hence, it is not possible to determine the domination number from the cardinality of an efficient dominating set, and the same is true for the total domination number and an efficient open dominating set. In this paper we introduce a restricted form of the efficient open domination that implies equal cardinality of all efficient total dominating sets in a digraph. We prove some general results on efficient open domination in digraphs and then we study efficient total domination in tournaments and directed tori.

2. Preliminaries

We consider only simple digraphs, i.e., having neither loops nor multiple arcs. Let G be a digraph with vertex set $V(G)$ and arc set $E(G)$. We say that vertex u *dominates* vertex v if $\overrightarrow{uv} \in E(G)$. The *open in-neighborhood* of a vertex v is the set $N^+(v)$ of all vertices that are dominating vertex v. (In other words, $N^+(v)$ is the set of all u's such that $\overrightarrow{uv} \in E(G)$.) Similarly, the *open out-neighborhood* of a vertex v is the set $N^-(v)$ of all vertices that are dominated by vertex v. The size of $N^+(v)$ is the *in-degree* of v, and the minimum $|N^+(u)|$, $u \in V(G)$, is the *minimum in-degree*, $\delta^+(G)$, of G. The *minimum out-degree*, $\delta^-(G)$, is defined analogously.

Closed neighborhoods of a vertex $v \in V(G)$ are $N^+[v] = N^+(v) \cup \{v\}$ and $N^-[v] = N^-(v) \cup \{v\}$. Set $D \subseteq V(G)$ is a *dominating set* of G if $\bigcup_{v\in D} N^-[v] = V(G)$, and $T \subseteq V(G)$ is a *total dominating set* of G if $\bigcup_{v\in T} N^-(v) = V(G)$. The *domination number* of G, $\gamma(G)$, is the minimum cardinality of a dominating set of G, while its *total domination number*, $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. Note that total dominating sets exist only in digraphs without sources (vertices of in-degree 0).

A set $P_{\text{in}} \subseteq V(G)$ is an *open in-packing* if $N^+(v) \cap N^+(u) = \emptyset$ for any two distinct vertices $u, v \in P_{\text{in}}$. An *open out-packing* is defined analogously. We say that digraph G is *efficiently open-dominated* if there exists an open in-packing $P_{\text{in}} \subseteq V(G)$ and an open out-packing $P_{\text{out}} \subseteq V(G)$, such that both $\{N^+(u);\, u \in P_{\text{in}}\}$ and $\{N^-(u);\, u \in P_{\text{out}}$ are partitions of $V(G)$. From the above definition it follows that if an open out-packing P_{out} is a partition of $V(G)$ then it is also a total dominating set, and each vertex in $V(G)$ is dominated by exactly one vertex from P_{out}. We call this set an *efficient total dominating set.* Similarly we define closed in-packing and closed out-packing of a digraph. We say that G is *efficiently closed-dominated* if there exists a closed in-packing $P_{\text{in}} \subseteq V(G)$ and a closed out-packing $P_{\text{out}} \subseteq V(G)$, such that both these packings are partitions of $V(G)$. If a closed out-packing P_{out} in G is a partition of $V(G)$ then it is also a dominating set, each vertex in $V(G) - P_{\text{out}}$ is dominated by exactly one vertex from P_{out} and P_{out} is an independent set. We call this set an *efficient dominating set.*

The existence of partitions of $V(G)$ to open in-neighborhoods and open out-neighborhoods implies that both $\delta^-(G)$ and $\delta^+(G)$ are positive numbers. In the rest of the paper we consider only digraphs without sources or sinks.

For the notions and notation not introduced here, see [5].

3. Properties of efficiently open-dominated digraphs.

Let G be a digraph. Its reverse digraph, G^-, is obtained by reversing all the arcs of G. In [13] it was proved that the difference between the total domination numbers of a digraph G and its reverse G^- can be arbitrary large even if G has an efficient total dominating set. It is also possible to construct a digraph with efficient total dominating sets of different cardinalities. Smallest example of such a digraph has 4 vertices $\{a,b,c,d\}$ and arcs $\overrightarrow{ab}$, $\overrightarrow{ac}$, $\overrightarrow{ad}$, $\overrightarrow{ba}$, $\overrightarrow{cb}$, $\overrightarrow{cd}$ and $\overrightarrow{dc}$, with both $\{a,b\}$ and $\{b,c,d\}$ being efficient total dominating sets. We show here that if a digraph is efficiently open-dominated then all efficient total dominating sets of G and G^- have the same size equal to $\gamma_t(G)$.

Lemma 3.1. *Let G be a digraph with a total dominating set T and an open in-packing P_{in}. Then $|P_{\text{in}}| \le |T|$.*

Proof. Let $v \in P_{\text{in}}$. As v is dominated by a vertex from T, the intersection $N^+(v) \cap T$ is not empty. As all in-neighborhoods $N^+(v)$, where $v \in P_{\text{in}}$, are disjoint, we have $|P_{\text{in}}| \le |T|$. $\square$

Theorem 3.2. *Let G be an efficiently open-dominated digraph. Then G^- is also efficiently open-dominated, and moreover, all efficient total dominating sets in G and G^- have the same cardinality $\gamma_t(G)$.*

Proof. Observe that the in-neighborhood of a vertex v in G is equal to the out-neighborhood of v in G^-. Hence, the partition of $V(G)$ to open in-neighborhoods is a partition of $V(G^-)$ to open out-neighborhoods and the partition of $V(G)$ to open out-neighborhoods is a partition of $V(G^-)$ to open in-neighborhoods, which gives the first part of the theorem.

Since G is efficiently open-dominated, there is an open in-packing P_{in} and an open out-packing P_{out} such that both P_{in} and P_{out} partition $V(G)$. Since P_{out} is a total dominating set in G, we have $|P_{\text{in}}| \le |P_{\text{out}}|$ by Lemma 3.1. On the other hand, P_{out} is an open in-packing in G^- and P_{in} is a total dominating set in G^-. Hence, $|P_{\text{out}}| \le |P_{\text{in}}|$ by Lemma 3.1, when applied to G^-. Thus, $|P_{\text{in}}| = |P_{\text{out}}|$.

Let T and T^- be efficient total dominating sets in G and G^-, respectively. Then T is an open out-packing in G, T^- is an open in-packing in G, and both T and T^- partition $V(G)$. As shown above, we have $|T| = |P_{\text{in}}|$ and $|T^-| = |P_{\text{out}}|$, so that $|T| = |T^-|$.

Finally, since for every total dominating set O in G we have $|P_{\text{in}}| \le |O|$ by Lemma 3.1, and since P_{out} is a total dominating set of cardinality $|P_{\text{in}}|$, we have $\gamma_t(G) = |P_{\text{in}}| = |P_{\text{out}}|$. $\square$

Let $U \subseteq V(G)$. By $S(U)$ we denote a subgraph of G induced by the vertex set U. We determine the structure of $S(T)$, where T is an efficient total dominating set in G.

Theorem 3.3. *Let G be a digraph with an efficient total dominating set T. Then each vertex $v \in T$ is either in a directed cycle in $S(T)$ or in a directed tree rooted at a vertex of a directed cycle in $S(T)$.*

Proof. Let v be a vertex in T. Since every vertex of G is dominated by exactly one vertex of T, the in-degree of each vertex in $S(T)$ is one. Denote by $P = (v_k, v_{k-1}, \ldots, v_1{=}v)$ a maximal path in $S(T)$, such that $\overrightarrow{v_{i+1}v_i} \in E(S(T))$. Since the in-degree of v_k is 1 in $S(T)$ and since the path P cannot be extended, we have $\overrightarrow{v_jv_k} \in E(S(T))$ for some j, $1 \le j < k$. Denote $Z = (v_j, v_k, v_{k-1}, \ldots, v_j)$. Then Z is a directed cycle. If $j = 1$ then v is in a directed cycle in $S(T)$. On the other hand if $j > 1$ then $(v_j, v_{j-1}, \dots, v_1{=}v)$ is a directed path rooted at the vertex v_j of directed cycle Z. $\square$

By Theorem 3.3, each connected component of a digraph induced by an efficient total dominating set consists of a directed cycle with possible pending rooted trees.

4. Tournaments.

A tournament is a digraph G, such that for every $u, v \in V(G)$, $u \neq v$, we have either $\overrightarrow{uv} \in E(G)$ or $\overrightarrow{vu} \in E(G)$. In this section we discuss the existence and properties of efficiently open-dominated tournaments. As in-degree and out-degree of each vertex should be positive, we consider only tournaments on at least 3 vertices.

Lemma 4.1. *Let G be a tournament with an efficient total dominating set T. Then $S(T)$ is a directed cycle of length* 3.

Proof. For every $u, v \in T$ either $\overrightarrow{uv} \in E(S(T))$ or $\overrightarrow{vu} \in E(S(T))$. By Theorem 3.3, every subgraph of $S(T)$ with t vertices has at most t arcs. However, if $|T| = t \geq 4$ then $S(T)$ has precisely $\binom{t}{2} > t$ arcs which contradicts Theorem 3.3. Hence, $|T| \leq 3$.

By Theorem 3.3, $S(T)$ has a directed cycle. As G has not a directed cycle of length 2, we have $|T| = 3$ and $S(T)$ is a directed cycle of length 3. $\square$

Theorem 4.2. *For any $n \geq 3$ there exists a tournament with n vertices that is efficiently open-dominated.*

Proof. First suppose that $n \leq 5$. Denote by G a tournament on 5 vertices $v_1, v_2, \dots, v_5$ with arcs $\overrightarrow{v_2v_1}$, $\overrightarrow{v_1v_3}$, $\overrightarrow{v_3v_2}$, $\overrightarrow{v_1v_4}$, $\overrightarrow{v_2v_4}$, $\overrightarrow{v_4v_3}$, $\overrightarrow{v_5v_1}$, $\overrightarrow{v_2v_5}$, $\overrightarrow{v_3v_5}$ and $\overrightarrow{v_5v_4}$. Now denote $G_n = S(\{v_1, v_2, \dots, v_n\})$, $3 \leq n \leq 5$. Then G_n is efficiently open-dominated tournament with in-packing $P_{\text{in}} = \{v_1, v_2, v_3\}$ and out-packing $P_{\text{out}} = \{v_{n-2}, v_{n-1}, v_n\}$, where both $\{N^+(v);\, v \in P_{\text{in}}\}$ and $\{N^-(v);\, v \in P_{\text{out}}\}$ are partitions of $V(G)$.

Now suppose that $n \geq 6$. By Lemma 4.1, it is enough to find a tournament G with six vertices a, b, c, A, B, C, such that both $(N^+(a), N^+(b), N^+(c))$ and $(N^-(A), N^-(B), N^-(C))$ are partitions of $V(G)$.

Partition the vertex set of G into three sets S_A, S_B, S_C, each with at least 2 vertices. Choose 2 vertices from S_A and denote them by A and a. Similarly, choose vertices B, b from S_B and C, c from S_C. Now we describe the orientation of arcs of the tournament.

1. We have $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CA} \in E(G)$.
2. For every $x \in (S_A - \{A\})$ we have $\overrightarrow{Ax} \in E(G)$. Analogously, for every $y \in (S_B - \{B\})$ and $z \in (S_C - \{C\})$ we have $\overrightarrow{By}, \overrightarrow{Cz} \in E(G)$.
3. For every $x \in (S_B - \{B\}) \cup (S_C - \{C\})$, $y \in (S_A - \{A\}) \cup (S_C - \{C\})$ and $z \in (S_A - \{A\}) \cup (S_B - \{B\})$, we have $\overrightarrow{xA}, \overrightarrow{yB}, \overrightarrow{zC} \in E(G)$.
4. For every $x \in (S_A - \{A\})$, $y \in (S_B - \{B\})$ and $z \in (S_C - \{C\})$, we have $\overrightarrow{xy}, \overrightarrow{yz}, \overrightarrow{zx} \in E(G)$.
5. For every $x \in (S_A - \{A, a\})$, $y \in (S_B - \{B, b\})$ and $z \in (S_C - \{C, c\})$, we have $\overrightarrow{ax}, \overrightarrow{by}, \overrightarrow{cz} \in E(G)$.
6. Orientation of edges between vertices (if any) inside the sets $(S_A - \{A, a\})$, $(S_B - \{B, b\})$ and $(S_C - \{C, c\})$ is arbitrary.

In the above construction, there are assigned orientations to all edges of the tournament. By 1, 2 and 3, we have $N^-(A) = (S_A - \{A\}) \cup \{B\}$, $N^-(B) = (S_B - \{B\}) \cup \{C\}$ and $N^-(C) = (S_C - \{C\}) \cup \{A\}$. Hence, $(N^-(A), N^-(B), N^-(C))$ is a partition of $V(G)$. By 3, 4 and 5, we have $N^+(a) = (S_C - \{C\}) \cup \{A\}$, $N^+(b) = (S_A - \{A\}) \cup \{B\}$ and $N^+(c) = (S_B - \{B\}) \cup \{C\}$. Hence, $(N^+(a), N^+(b), N^+(c))$ is a partition of $V(G)$. As a consequence, $\{A, B, C\}$ is an efficient total dominating set in G and $\{a, b, c\}$ is an efficient total dominating set in G^-. $\square$

5. Efficient open-domination in directed tori.

The Cartesian product of two directed graphs G_1 and G_2 is a directed graph $G_1 \square G_2$ with vertex set $V(G_1) \times V(G_2)$. For two vertices $[x_1, x_2], [y_1, y_2] \in V(G_1) \times V(G_2)$, there is an arc from $[x_1, x_2]$ to $[y_1, y_2]$ if and only if either $x_1 = y_1$ and $\overrightarrow{x_2 y_2} \in E(G_2)$ or $x_2 = y_2$ and $\overrightarrow{x_1 y_1} \in E(G_1)$. As the directed cycle $\overrightarrow{C_k}$ is isomorphic to a Cayley graph of additive group $\mathbb{Z}_k$ with generator 1, we shall denote its vertices by $V(\overrightarrow{C_k}) = \{0, 1, \ldots, k-1\}$. The Cartesian product of n directed graphs, $n \geq 3$, is defined in a similar way. However, to simplify the notation we omit interior brackets in that case. The n-dimensional directed torus is $T(k_1, k_2, ..., k_n) = \overrightarrow{C_{k_1}} \square \overrightarrow{C_{k_2}} \square \ldots \square \overrightarrow{C_{k_n}}$, where $k_i \geq 2$ for $1 = 1, 2, \ldots, n$.

First we show that if a digraph G has an efficient dominating set then the Cartesian product of G and a directed cycle has an efficient total dominating set.

Lemma 5.1. *Let G be a digraph with an efficient dominating set. Then there exists an efficient total dominating set in $G \square \overrightarrow{C_k}$.*

Proof. Let D be an efficient dominating set in G. Then $\bigcup_{v \in D} N^-[v] = V(G)$ and all closed out-neighborhoods $N^-[v]$ are disjoint for $v \in D$. We show that $D' = D \times V(\overrightarrow{C_k})$ is an efficient total dominating set in $G \square \overrightarrow{C_k}$. Let $[u, t] \in G \square \overrightarrow{C_k}$. We distinguish two cases:

1. $u \in D$. As D is an efficient dominating set in G, there is no $v \in D$ such that $\overrightarrow{vu} \in E(G)$. Hence, $[u, t-1]$ is the unique vertex of D' dominating $[u, t]$.
2. $u \notin D$. As D is an efficient dominating set in G, there is a unique vertex $v \in D$ such that $\overrightarrow{vu} \in E(G)$. Since $[u, t-1] \notin D'$, $[v, t]$ is the unique vertex of D' dominating $[u, t]$.

As every vertex of $G \square \overrightarrow{C_k}$ is dominated by precisely one vertex of D', the set D' is an efficient total dominating set in $G \square \overrightarrow{C_k}$. $\square$

Each vertex in the n-dimensional directed torus has in-degree and out-degree equal to n. So if the n-dimensional torus is efficiently open dominated then the cardinality of its vertex set is divisible by n. The following result was proved in [8].

Lemma 5.2. *For any positive integers $t_1, t_2, \ldots, t_n$, the n-dimensional directed torus $T((n+1)t_1, (n+1)t_2, \ldots, (n+1)t_n)$ has an efficient dominating set.*

As n-dimensional torus $T(k_1, k_2, ..., k_n)$ is a Cartesian product of $n-1$ dimensional torus and a directed cycle, we can formulate a sufficient condition for a directed torus to be efficiently open-dominated.

Theorem 5.3. *Let k_i be divisible by n for all i, $1 \leq i \leq n-1$. Then the directed torus $T(k_1, k_2, \ldots, k_n)$ is efficiently open-dominated.*

Proof. By Lemma 5.1 and Lemma 5.2, $T(k_1, k_2, ..., k_n)$ has an efficient total dominating set. Hence, its vertex set has a partition to open out-neighborhoods. As a directed torus is isomorphic to its reverse, it has also a partition to open in-neighborhoods and the result follows. $\square$

In the proof of Lemma 5.1, there was constructed an efficient total dominating set T, such that $S(T)$ is a disjoint union of directed cycles. We conclude the paper by a theorem which states that all efficient total dominating sets in a directed torus are of this type.

Theorem 5.4. *Let $T(k_1, k_2, \dots, k_n)$ be an efficiently open-dominated directed torus and let T be an efficient total dominating set in G. Then each connected component of $S(T)$ is a directed cycle.*

Proof. By Theorem 3.3, each connected component of $S(T)$ is a directed cycle with possible directed trees rooted at its vertices. To prove the statement of the theorem, it is enough to show that the out-degree of each vertex $u \in T$ is at most 1 in $S(T)$.

By way of contradiction, suppose that in $S(T)$ there exists a vertex $u = (x_1, x_2, \dots, x_n)$ with at least 2 successors $v_1 = (x_1, \dots, x_i{+}1, \dots, x_n)$ and $v_2 = (x_1, \dots, x_j{+}1, \dots, x_n)$ for some indices $i < j$, where the addition is taken modulo k_i and k_j, respectively. As both v_1 and v_2 dominate $(x_1, \dots, x_i{+}1, \dots, x_j + 1, \dots, x_n)$, the set T is not an efficient total dominating set, a contradiction. Hence, each vertex in $S(T)$ has out-degree at most 1. $\square$

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