A lower bound for the number of orientable triangular embeddings of some complete graphs.

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Abstract

We prove that, for a certain positive constant a and for an infinite set of values of n, the number of nonisomorphic face 2-colourable triangular embeddings of the complete graph K_n in an orientable surface is at least n^{an^2}.

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1 Introduction

As we explain below, the number of nonisomorphic minimum genus (either orientable or nonorientable) embeddings of the complete graph K_n cannot exceed $n^{n^2(\frac{1}{3}-o(1))}$ as $n \to \infty$. Until recently, the best known lower bound on this number was $2^{n^2(a-o(1))}$ for a certain infinite set of values n and positive constant a [2]. Also, for all sufficiently large n, V. P. Korzhik and H.-J. Voss have established a lower bound of the form 2^{an} [13, 14, 15, 16]. Then in [5] it was shown that for a certain positive constant a and for an infinite set of values n, the number of nonisomorphic minimum genus embeddings of the complete graph K_n is at least n^{an^2}. However all the embeddings of K_n in [5] were in a nonorientable surface. In the current paper we establish an analogous result for embeddings of K_n in an orientable surface. The method used is similar to that of [5] where the first step was to construct a special face 2-colourable triangular embedding of the complete regular tripartite graph $K_{n,n,n}$ for certain even values of n. Here the first step is to do a similar job for certain odd values of n.

For general background material on topological embeddings, we refer the reader to [10] and [17]. Our embeddings will always be in closed connected 2-manifolds without a boundary. A graph embedding is *face 2-colourable* if the faces may be coloured in such a way that any two faces with a common boundary receive different colours. We will always take the colours to be black and white. A graph embedding is called *triangular* if all the faces are triangles. It was shown in [6] that a triangular embedding of $K_{n,n,n}$ is face 2-colourable if and only if the supporting surface is orientable, and the surface is therefore a sphere with an appropriate number of handles.

A triangular embedding of K_n determines a *twofold triple system*, TTS(n); this comprises a pair $(V, \mathcal{B})$ where V is an n-element set (the *points*) and $\mathcal{B}$ is a collection (that is, a multiset) of 3-element subsets (the *blocks* or *triples*) of V such that each pair of elements from V is contained in precisely two blocks. It is well known that an TTS(n) exists if and only if $n \equiv 0$ or 1 (mod 3) [3]. The vertices of the embedded graph K_n form the points of the design and the faces form the triples. As has been remarked elsewhere, for example in [4], an upper bound on the number of triangular embeddings of K_n is easily provided by an upper bound on the number of TTS(n)s. In brief, this bound may be established as follows. Take any TTS(n) $= (V, \mathcal{B}_0)$ where $V = \{1, 2, \ldots, n\}$. Let B_1 be the block $\{1, 2, a_1\}$, where a_1 is the minimal value such that $\{1, 2, a_1\} \in \mathcal{B}_0$, and then put $\mathcal{B}_1 = \mathcal{B}_0 \setminus \{B_1\}$. We iterate this process: at stage k let $\{i, j\}$ be the lowest pair that appears in $\mathcal{B}_{k-1}$, let a_k be the minimal point for which $B_k = \{i, j, a_k\} \in \mathcal{B}_{k-1}$, and then put $\mathcal{B}_k = \mathcal{B}_{k-1} \setminus \{B_k\}$. Stop when $\mathcal{B}_k$ is empty. The ordered list $a_1, a_2, ..., a_b$, where $b = n(n-1)/3$ is the number of triples in $\mathcal{B}_0$, determines the TTS(n). There are at most $(n-2)^{n(n-1)/3}$ such lists, so there is at most this number of triangular embeddings of K_n. For minimum genus embeddings which are not triangular, there is a small number of non-triangular faces but the number and size of these faces is bounded by a (small) constant. These faces can be listed separately and add at most a small constant to the exponent, yielding an upper bound $n^{n^2(\frac{1}{3}-o(1))}$ for the number of distinct minimum genus embeddings of K_n with vertex set $\{1, 2, \ldots, n\}$. The maximum size of an isomorphism class of such embeddings is $n!$, but this has smaller order of magnitude than $n^{n^2(\frac{1}{3}-o(1))}$, so we may also take the latter expression as an upper bound for the number of nonisomorphic minimum genus embeddings of K_n.

When a triangular embedding of K_n is face 2-colourable, the associated TTS(n) partitions into two *Steiner triple systems*, STS(n), one for each colour class. An STS(n) comprises a pair $(V, \mathcal{B})$ where V is an n-element set (the *points*) and $\mathcal{B}$ is a set of 3-element subsets (the *blocks* or *triples*) of V such that each pair of elements from V is contained in precisely one block. It is well known that an STS(n) exists if and only if $n \equiv 1$ or 3 (mod 6) [12]. We say that two STS(n)s are *biembeddable* in a surface if there is a face 2-colourable triangular embedding of K_n in which the face sets forming the two colour classes give isomorphic copies of the two systems. Apart from the small cases $v = 3$ and $v = 7$, there are no known examples of a pair of Steiner triple systems which cannot be biembedded in a nonorientable surface. A necessary condition for the existence of an orientable biembedding of two STS(n)s is that $n \equiv 3$ or 7 (mod 12). However, it was shown in [8] that many pairs of STS(15)s admit no orientable biembeddings.

A face 2-colourable triangular embedding of $K_{n,n,n}$ determines two *transversal designs*, TD$(3, n)$, one for each colour class. Such a design comprises a triple $(V, \mathcal{G}, \mathcal{B})$, where V is a $3n$-element set (the *points*), $\mathcal{G}$ is a partition of V into three disjoint sets (the *groups*) each of cardinality n, and $\mathcal{B}$ is a set of 3-element subsets of V (the *triples*), such that each pair of elements from V is either contained in precisely one triple or one group, but not both. The vertices of the embedded graph $K_{n,n,n}$ form the points of each design, the tripartition determines the groups, and the faces in each colour class form the triples of each design.

A $\mathrm{TD}(3,n)$ determines a Latin square of order n by assigning the three groups of the design as labels for the rows, columns and entries (in any one of six possible orders) of the Latin square. Conversely any Latin square of order n determines a $\mathrm{TD}(3,n)$. Two Latin squares are said to be in the same *main class* or *paratopic* if the corresponding $\mathrm{TD}(3,n)$s are isomorphic. Thus a face 2-colourable triangular embedding of $K_{n,n,n}$ may be considered as a biembedding of two $\mathrm{TD}(3,n)$s or, equivalently, two Latin squares. To be precise, we say that two Latin squares of order n are *biembeddable* in a surface if there is a face 2-colourable triangular embedding of $K_{n,n,n}$ in which the face sets forming the two colour classes give paratopic copies of the two squares. By considering bounds on the number of Latin squares of order n, it is easy to see that the number of nonisomorphic biembeddings of such squares cannot exceed $n^{2n^2(1-o(1))}$. A necessary condition for two Latin squares of order n to be biembeddable was established in [11] and, as a consequence, it is clear that infinitely many pairs of Latin squares admit no biembeddings.

A *parallel class* of triples in a $\mathrm{TD}(3,n)$ is a set of triples in which each point of the design appears precisely once. Such a parallel class is equivalent to a transversal in a corresponding Latin square. In combinatorial design terminology, a *Pasch configuration* is a set of four triples on six points having the form $\{\{a,b,c\},\{a,d,e\},\{b,d,f\},\{c,e,f\}\}$. In a Latin square, such a configuration corresponds to a subsquare of order 2, which we will call a 2-subsquare.

2 Complete tripartite graphs

We start by recalling, in a slightly modified form, a definition from [5].

Definition 2.1 *Suppose that $A = (a_{i,j})$ is an $n \times n$ array with rows and columns indexed by a set $M = \{m_1, m_2, \ldots, m_n\}$. If, for each $i \in \{1, 2, \ldots, n\}$, the permutation*

$$\begin{pmatrix} a_{m_i,m_1} & a_{m_i,m_2} & \cdots & a_{m_i,m_n} \\ a_{m_{i+1},m_1} & a_{m_{i+1},m_2} & \cdots & a_{m_{i+1},m_n} \end{pmatrix}$$

is a single cycle of length n, including the case $i = n$ when $i+1$ is taken as 1, then we will say that A is consecutively row Hamiltonian*, cr-Hamiltonian for short.*

The following result was proved in [5].

Lemma 2.1 *Suppose that $A = (a_{i,j})$ is a cr-Hamiltonian Latin square of order n. Then A has a biembedding with a copy of itself.*

Our key lemma may now be stated.

Lemma 2.2 *For each $m = 2^{2t+1} - 1$ with $t \geq 1$ there is a cr-Hamiltonian Latin square of order $3m$ that has $9m(m-1)(m-3)/4$ 2-subsquares and a transversal.*

Proof. Since $2^{2t+1} \equiv 2 \pmod 3$, m is not divisible by 3. Now let x be a primitive element of the Galois field $\mathrm{GF}(2^{2t+1})$. Then x, and hence x^3, both have order m. Consider the Latin square L formed by the Cayley table of the Steiner quasigroup corresponding to the projective STS(m). This may have its rows and columns indexed by $\{1, x, x^2, \ldots, x^{m-1}\}$. The entry in row x^i and column x^j is $x^i + x^j$ when $i \neq j$, and x^i when $i = j$. So we have

$$L = \begin{array}{r|cccccc}
 & 1 & x & x^2 & x^3 & \ldots & x^{m-1} \\ \hline
1 & 1 & 1+x & 1+x^2 & 1+x^3 & \ldots & 1+x^{m-1} \\
x & 1+x & x & x+x^2 & x+x^3 & \ldots & x+x^{m-1} \\
x^2 & 1+x^2 & x+x^2 & x^2 & x^2+x^3 & \ldots & x^2+x^{m-1} \\
x^3 & 1+x^3 & x+x^3 & x^2+x^3 & x^3 & \ldots & x^3+x^{m-1} \\
\vdots & & & & & \vdots & \\
x^{m-1} & 1+x^{m-1} & x+x^{m-1} & x^2+x^{m-1} & x^3+x^{m-1} & \ldots & x^{m-1}
\end{array}$$

Note that L has a transversal on the leading diagonal. We need to determine the number of 2-subsquares in L. To do this, choose any row r_1 and any column c_1, where $c_1 \neq r_1$. The number of such choices is $m(m-1)$. Suppose that the entry in row r_1, column c_1 is e_1, so that $r_1 + c_1 = e_1$. Then choose any column $c_2 \neq r_1, c_1, e_1$. There are $m - 3$ such choices for c_2. Suppose that the entry in row r_1, column c_2 is e_2 so that $r_1 + c_2 = e_2$. Now locate the entry e_2 in column c_1; suppose this occurs in row r_2. Then $r_2 \neq r_1$ and $r_2 + c_1 = e_2$. It follows from the three equations that $r_2 + c_2 = e_1$. Thus there is a 2-subsquare on the positions $(r_1, c_1), (r_1, c_2), (r_2, c_1), (r_2, c_2)$. Since every such 2-subsquare will be counted four times in this enumeration, it follows that L contains at least $m(m-1)(m-3)/4$ 2-subsquares. However, any other 2-subsquare would have to contain a cell from the leading diagonal and it is easy to see that this is impossible. Hence L contains precisely $m(m-1)(m-3)/4$ 2-subsquares.

By defining $\alpha_i = 1 + x^i$ for $i = 1, 2, \ldots, m-1$ and $\alpha_0 = 1$, we may write the first row of entries in L as $\mathbf{a}_0 = (\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$. Then if $\mathbf{a}_i$ is defined for $i = 1, 2, \ldots m-1$ by $\mathbf{a}_i = (\alpha_{m-i}, \alpha_{m-i+1}, \ldots, \alpha_{m-i-1})$, with subscript arithmetic modulo m, the x^i row of L is just $x^i\mathbf{a}_i$. Now take the $3m \times 3m$ array A with entries $\alpha_i, \alpha'_i, \alpha''_i$ for $i = 0, 1, \ldots, m-1$ and given by

$$A = \left(\begin{array}{c|c|c}
\mathbf{a}_0 & (\mathbf{a}_0)' & (\mathbf{a}_0)'' \\
(\mathbf{a}_0)' & (\mathbf{a}_1)'' & \mathbf{a}_1 \\
(\mathbf{a}_1)'' & \mathbf{a}_1 & (\mathbf{a}_1)' \\ \hline
x\mathbf{a}_1 & (x\mathbf{a}_1)' & (x\mathbf{a}_1)'' \\
(x\mathbf{a}_1)' & (x\mathbf{a}_2)'' & x\mathbf{a}_2 \\
(x\mathbf{a}_2)'' & x\mathbf{a}_2 & (x\mathbf{a}_2)' \\ \hline
x^2\mathbf{a}_2 & (x^2\mathbf{a}_2)' & (x^2\mathbf{a}_2)'' \\
(x^2\mathbf{a}_2)' & (x^2\mathbf{a}_3)'' & x^2\mathbf{a}_3 \\
(x^2\mathbf{a}_3)'' & x^2\mathbf{a}_3 & (x^2\mathbf{a}_3)' \\ \hline
\vdots & \vdots & \vdots \\ \hline
x^{m-1}\mathbf{a}_{m-1} & (x^{m-1}\mathbf{a}_{m-1})' & (x^{m-1}\mathbf{a}_{m-1})'' \\
(x^{m-1}\mathbf{a}_{m-1})' & (x^{m-1}\mathbf{a}_0)'' & x^{m-1}\mathbf{a}_0 \\
(x^{m-1}\mathbf{a}_0)'' & x^{m-1}\mathbf{a}_0 & (x^{m-1}\mathbf{a}_0)'
\end{array}\right).$$

Here, for example, $(x\mathbf{a}_1)'$ denotes the vector $(x'\alpha'_{m-1}, x'\alpha'_0, x'\alpha'_1, \ldots, x'\alpha'_{m-2})$ $= ((1+x)', x', (x+x^2)' \ldots, (x+x^{m-1})')$. We show that A has the desired properties.

First we prove that A is a cr-Hamiltonian array. To do this, number the rows from 0 to $3m-1$ and denote by π_j the permutation formed from rows j and $j+1$. Then for $i = 0, 1, \ldots, m-1$,

$$\begin{aligned}
\pi_{3i} &= \begin{pmatrix} x^i\mathbf{a}_i & (x^i\mathbf{a}_i)' & (x^i\mathbf{a}_i)'' \\ (x^i\mathbf{a}_i)' & (x^i\mathbf{a}_{i+1})'' & x^i\mathbf{a}_{i+1} \end{pmatrix} \\
&= (x^i\alpha_{m-i}, (x^i\alpha_{m-i})', (x^i\alpha_{m-i-1})'', x^i\alpha_{m-i-2}, \ldots, (x^i\alpha_{m-i+1})''),
\end{aligned}$$

which is a cycle of length $3m$ because m is odd. Similarly

$$\pi_{3i+1} = ((x^i\alpha_{m-i})', (x^i\alpha_{m-i-1})'', x^i\alpha_{m-i-1}, (x^i\alpha_{m-i-1})', \ldots, x^i\alpha_{m-i})$$

is a cycle of length $3m$ and

$$\begin{aligned}
\pi_{3i+2} = ((x^i\alpha_{m-i-1})'', x^{i+1}\alpha_{m-i-1}, (x^{i+2}\alpha_{m-i-1})', (x^{i+3}\alpha_{m-i-1})'', \\
\ldots, (x^{i-1}\alpha_{m-i-1})')
\end{aligned}$$

is also a cycle of length $3m$ because x^3 has order m. Hence A is cr-Hamiltonian.

To establish the remaining properties we permute the rows of A in an obvious fashion to obtain the following array:

$$A^* = \left(\begin{array}{c|c|c} \mathbf{a}_0 & (\mathbf{a}_0)' & (\mathbf{a}_0)'' \\ x\mathbf{a}_1 & (x\mathbf{a}_1)' & (x\mathbf{a}_1)'' \\ x^2\mathbf{a}_2 & (x^2\mathbf{a}_2)' & (x^2\mathbf{a}_2)'' \\ \vdots & \vdots & \vdots \\ x^{m-1}\mathbf{a}_{m-1} & (x^{m-1}\mathbf{a}_{m-1})' & (x^{m-1}\mathbf{a}_{m-1})'' \\ \hline (\mathbf{a}_0)' & (\mathbf{a}_1)'' & \mathbf{a}_1 \\ (x\mathbf{a}_1)' & (x\mathbf{a}_2)'' & x\mathbf{a}_2 \\ (x^2\mathbf{a}_2)' & (x^2\mathbf{a}_3)'' & x^2\mathbf{a}_3 \\ \vdots & \vdots & \vdots \\ (x^{m-1}\mathbf{a}_{m-1})' & (x^{m-1}\mathbf{a}_0)'' & x^{m-1}\mathbf{a}_0 \\ \hline (\mathbf{a}_1)'' & \mathbf{a}_1 & (\mathbf{a}_1)' \\ (x\mathbf{a}_2)'' & x\mathbf{a}_2 & (x\mathbf{a}_2)' \\ (x^2\mathbf{a}_3)'' & x^2\mathbf{a}_3 & (x^2\mathbf{a}_3)' \\ \vdots & \vdots & \vdots \\ (x^{m-1}\mathbf{a}_0)'' & x^{m-1}\mathbf{a}_0 & (x^{m-1}\mathbf{a}_0)' \end{array}\right) = \left(\begin{array}{c|c|c} L & L' & L'' \\ \hline L' & M'' & M \\ \hline M'' & M & M' \end{array}\right).$$

Here L is just the original Latin square formed by the Cayley table of the Steiner quasigroup corresponding to the projective STS(m), and L', L'' are copies of L. The square M is obtained from L by multiplying all entries by x^{m-1} and permuting the rows, and M', M'' are copies of M. It follows that A^* is a Latin square and that it has 9 times the number of 2-subsquares that L has. Also, since L has a transversal, so also does M, and consequently so does A^*. Hence A itself is a Latin square that has $9m(m-1)(m-3)/4$ 2-subsquares and a transversal. □

Applying the result of Lemma 2.1 to the square A of order $3m = 3(2^{2t+1} - 1)$ constructed in Lemma 2.2 gives a biembedding of that Latin square. The biembedding is actually a face 2-colourable triangular embedding of $K_{3m,3m,3m}$ in an orientable surface. In this embedding, each 2-subsquare in each colour class is realized as a Pasch configuration comprising four triangles $(a, b, c), (a, d, e), (f, b, e), (f, d, c)$, where a, f are row labels, b, d are column labels and e, c are entries. There will be $9m(m-1)(m-3)/4$ Pasch configurations in each colour class. The transversal in A will be realized as a parallel class of $3m$ triangles in each of the two colour classes. We now apply a recursive construction for biembeddings of Latin squares with this biembedding as one of the ingredients. The result is a large number of nonisomorphic biembeddings. Our basic construction is a generalization of one given in [9] which is described in more detail in [5]. We now give a brief description of that construction.

Take a face 2-colourable triangular embedding P of $K_{p,p,p}$ and another Q of $K_{q,q,q}$, and assume that the latter has a parallel class of triangular faces in one of the two colour classes, say black. Now take q copies of P, say P_i for $i = 1, 2, \dots, q$. For each oriented white triangular face $W = (a, b, c)$ of P, we "bridge" the corresponding white triangles $W_i = (a_i, b_i, c_i)$ of the embeddings P_i. To do this we first cut out the white triangular face W_i from each P_i. Then we take a copy of Q and label the vertex parts with $\{a_i\}, \{b_i\}, \{c_i\}$ in such a way that the parallel class has oriented black triangles labelled (a_i, c_i, b_i). Next we cut out each of the black triangular faces (a_i, c_i, b_i) from Q. Then for each i we glue the boundaries of the white triangle (a_i, b_i, c_i) of P_i to the boundaries of the black triangle (a_i, c_i, b_i) of Q so that corresponding vertices and edges are identified. Repeating this process for every white triangle of P results in a face 2-colourable triangular embedding of $K_{pq,pq,pq}$ in an orientable surface.

In this construction, the bridging operation provides all the "missing" adjacencies between the q copies P_i. The bridge across the q triangles $W_i = (a_i, b_i, c_i)$ yields the adjacencies a_ib_j, a_ic_j, b_ic_j for $i, j = 1, 2, \dots, q$, $i \neq j$. Now suppose that P contains a Pasch configuration $(a, b, c), (a, d, e), (f, b, e), (f, d, c)$. The four corresponding bridges provide the missing adjacencies $a_ib_j, a_ic_j, b_ic_j, a_id_j, a_ie_j, d_ie_j, f_ib_j, f_ie_j, b_ie_j, f_id_j, f_ic_j, d_ic_j$ for $i \neq j$. It is, however, possible to provide these adjacencies by an alternative arrangement of bridges that singles out one embedding, say P_1, for special treatment. For $2 \leq i \leq q$ we bridge (a_1, d_1, e_1) to (a_i, b_i, c_i), (a_1, b_1, c_1) to (a_i, d_i, e_i), (f_1, d_1, c_1) to (f_i, b_i, e_i), and (f_1, b_1, e_1) to (f_i, d_i, c_i) by suitable renaming of the vertices of the four bridges involved. For example, one of the four copies of Q originally had its parallel class of black triangles labelled (a_i, c_i, b_i) for $1 \leq i \leq q$ but now we relabel (a_1, c_1, b_1) as (a_1, e_1, d_1), and we do a similar relabelling of the other three copies of Q. For $i, j \geq 2$ and $i \neq j$ this leaves all the adjacencies generated by the original bridges between P_i and P_j unaltered. However, for $i \geq 2$, the first bridge then provides the adjacencies $a_1b_i, a_1c_i, d_1a_i, d_1c_i, e_1a_i, e_1b_i$, the second provides $a_1d_i, a_1e_i, b_1a_i, b_1e_i, c_1a_i, c_1d_i$, the third provides f_1b_i, f_1e_i, $d_1f_i, d_1e_i, c_1f_i, c_1b_i$, and the fourth provides $f_1d_i, f_1c_i, b_1f_i, b_1c_i, e_1f_i, e_1d_i$. For each $i \geq 2$, these 24 adjacencies are the same as the 24 adjacencies arising for the original bridging arrangement across the Pasch configuration between P_1 and P_i.

We will call the original arrangement of bridges *standard* and an arrangement of the type just described *non-standard*. This is a slightly tighter definition of the term than was used in [5] but it will suffice for our purposes here. We next recall another definition from [5].

Definition 2.2 *Given a Latin square A, a collection $\mathcal{C}$ of 2-subsquares will be called an* independent collection of 2-subsquares *if no two of the 2-subsquares have a common (row, column, entry) triple.*

Since 2-subsquares in a Latin square A correspond to Pasch configurations in any biembedding of A, we will also use the term *independent collection of Pasch configurations.*

Lemma 2.3 *The Latin square A of order $p = 3(2^{2t+1} - 1)$ defined in Lemma 2.2 has at least $(\frac{4p-33}{3})^{(\frac{p(p-3)}{16}-1)}$ independent collections of 2-subsquares.*

Proof. Put $m = (2^{2t+1} - 1)$ and $s = 9m(m-1)(m-3)/4$, the number of 2-subsquares in A. Denote by I_k the number of distinct independent collections of 2-subsquares in A that contain precisely k 2-subsquares. Each off-diagonal (row, column, entry) triple of L can lie in at most $(m-3)$ 2-subsquares of L. Hence each triple of A that lies in a 2-subsquare of A can lie in at most $(m-3)$ 2-subsquares of A. Since there are four triples to each 2-subsquare, it follows that each 2-subsquare of A can intersect at most $4(m-3)$ 2-subsquares of A in a common triple. So, for $k - 1 < \frac{s}{4(m-3)}$ we have

$$\begin{aligned} I_k &\geq s(s-4(m-3))(s-8(m-3))\cdots(s-4(k-1)(m-3))/k! \\ &\geq (4(m-3))^k N(N-1)(N-2)\cdots(N-(k-1))/k! \end{aligned}$$

where $N = \lfloor \frac{s}{4(m-3)} \rfloor$. Then, summing over $k = 0, 1, \ldots, N$ gives the number of distinct independent collections of 2-subsquares as at least

$$(1 + 4(m-3))^N \geq (4m-11)^{(\frac{s}{4(m-3)}-1)} = \left(\frac{4p-33}{3}\right)^{(\frac{p(p-3)}{16}-1)}. \qquad \square$$

The following lemma was also established in [5]. The proof is simplified slightly by our tighter definition of "non-standard".

Lemma 2.4 *Suppose that M is a face 2-colourable triangular embedding of $K_{p,p,p}$ and that $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$ are two different independent collections of Pasch configurations in the same colour class, say white. For $i = 1, 2$, let $M^{(i)}$ be the embedding that results when we apply standard $K_{q,q,q}$ bridges to each white face not in $\mathcal{C}^{(i)}$ and non-standard bridge arrangements to each Pasch configuration in $\mathcal{C}^{(i)}$. Then the embeddings $M^{(1)}$ and $M^{(2)}$ will be differently labelled.*

Proof. If the Pasch configurations in $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$ do not cover the same triples, then there is a Pasch configuration $\{(a,b,c),(a,d,e),(f,b,e),(f,d,c)\}$ in $\mathcal{C}^{(1)}$, and without loss of generality we may assume that (a,b,c) does not lie in any Pasch configuration in $\mathcal{C}^{(2)}$. Then $M^{(1)}$ has a white triangle (a_1,b_1,e_i) for some $i \neq 1$. However, in $M^{(2)}$ the edge a_1b_1 lies in the white triangle (a_1,b_1,c_i). So in this case $M^{(1)}$ and $M^{(2)}$ are differently labelled.

Now suppose that $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$ cover the same triples and that $M^{(1)}$ and $M^{(2)}$ are identically labelled. We derive a contradiction as follows. If $\mathcal{C}^{(1)}$ contains the Pasch configuration $R = \{(a,b,c),(a,d,e),(f,b,e),(f,d,c)\}$ then we may suppose without loss of generality that the contributing blocks do not lie in the same Pasch configuration in $\mathcal{C}^{(2)}$ and that the non-standard bridging arrangement for $\mathcal{C}^{(1)}$ contains a white triangle (a_1,b_1,e_i) with $i \neq 1$. Since the same triangle arises in the non-standard bridging arrangement for $\mathcal{C}^{(2)}$, the triple (a,b,c) and either (a,d,e) or (f,b,e) must lie together in a Pasch configuration in $\mathcal{C}^{(2)}$. So, ignoring for a moment the tripartition which orders entries in the triples, $\mathcal{C}^{(2)}$ contains either the Pasch configuration $S = \{\{a,b,c\},\{a,d,e\},\{g,b,d\},\{g,c,e\}\}$ or $T = \{\{a,b,c\},\{f,b,e\},\{h,a,f\},\{h,c,e\}\}$ for some suitable g or $h \in V(K_{p,p,p})$. However, there is no edge ce in the tripartite graph $K_{p,p,p}$, and so neither S nor T can exist. It follows that $M^{(1)}$ and $M^{(2)}$ must be differently labelled. □

As previously remarked, an independent collection of Pasch configurations in a colour class corresponds exactly to an independent collection of 2-subsquares in the associated Latin square. Furthermore, a $K_{q,q,q}$ bridge is simply a face 2-colourable triangular embedding of the graph having a parallel class of faces in one of the two colour classes. The embeddings $M^{(1)}$ and $M^{(2)}$ which result are face 2-colourable triangular embeddings of $K_{pq,pq,pq}$. In addition, if the black faces of M have a parallel class then the black faces of the q copies M, all of which are unaltered by bridging the white triangles, have a common parallel class in both the embeddings $M^{(1)}$ and $M^{(2)}$.

Consequently, we may state the following theorem.

Theorem 2.1 *Let $p = 3(2^{2t+1} - 1)$ where $t \geq 1$ and suppose there is a face 2-colourable triangular embedding of $K_{q,q,q}$ having a parallel class in one of the colour classes, say black. Then there are at least*

$$\left(\frac{4p-33}{3}\right)^{(\frac{p(p-3)}{16}-1)}$$

differently labelled face 2-colourable triangular embeddings of $K_{pq,pq,pq}$ all of which have a common parallel class of black triangular faces. Furthermore, there are at least

$$\frac{\left(\frac{4p-33}{3}\right)^{(\frac{p(p-3)}{16}-1)}}{6((pq)!)^3}$$

nonisomorphic face 2-colourable triangular embeddings of $K_{pq,pq,pq}$.

Proof. The first part follows from the preceding discussion. The second part follows because the maximum possible size of an isomorphism class of such an embedding is $6((pq)!)^3$. □

Note that for odd $q > 1$, there is a face 2-colourable triangular embedding of $K_{q,q,q}$ given by the Latin squares with entries $C(i,j) = i + j \pmod q$ and $D(i,j) = i + j + 1 \pmod q$, and that each of these squares has a transversal. Using this with $q = 3$, we have the following corollary.

Corollary 2.1.1 *For $n = 9(2^{2t+1}-1)$ as $t \to \infty$, there are at least $n^{\frac{n^2}{144}(1-o(1))}$ nonisomorphic face 2-colourable triangular embeddings of $K_{n,n,n}$, each of which has a parallel class in one colour.*

Proof. In Theorem 2.1 take $q = 3$ and replace p by $n/3$. The number of isomorphism classes of the resulting $K_{n,n,n}$ embeddings is then at least

$$\frac{\left(\frac{4n-99}{9}\right)^{(\frac{n(n-9)}{144}-1)}}{6(n!)^3} = n^{\frac{n^2}{144}(1-o(1))}$$

□

Re-phrasing the corollary in the language of Latin squares, it follows that for $n = 9(2^{2t+1} - 1)$ as $t \to \infty$, there are at least $n^{\frac{n^2}{144}(1-o(1))}$ nonisomorphic biembeddings of Latin squares of order n, each of which has a parallel class in one colour. Note that these biembeddings are necessarily in an orientable surface.

3 Complete graphs

A recursive construction given in [9] forms the basis for the results of this section and we start by describing it informally.

The construction takes a face 2-colourable triangular embedding M of K_m, a face 2-colourable triangular embedding R of $K_{r,r,r}$ having a parallel class of black triangular faces, and a face 2-colourable triangular embedding S of K_{2r+1}. The embeddings M and S are biembeddings of Steiner triple systems, and the embedding R is a biembedding of Latin squares. Choose any single vertex ∞ of the embedding M and delete the cap at this point; that is to say, remove all the triangular faces incident with ∞ from M to leave a triangular embedding $\bar{M}$ of K_{m-1} in a bordered surface. Next take r copies of $\bar{M}$, say $\bar{M}_i$ for $i = 1, 2, \ldots, r$. For each white triangular face $W = (a, b, c)$ of $\bar{M}$, bridge the corresponding white triangles $W_i = (a_i, b_i, c_i)$ of the embeddings $\bar{M}_i$ in the manner described in the previous section, using a copy of the embedding R. When this operation is complete, the resulting embedding is "close" to that of $K_{r(m-1)}$ but with a small number of missing adjacencies corresponding to the deleted triangles, and the supporting surface has r disjoint boundaries. If $(m-1)/2$ is odd, these boundaries can then be capped by means of an auxiliary embedding A, which provides one extra vertex and all missing adjacencies, to give a face 2-colourable triangular embedding of $K_{r(m-1)+1}$. The embedding A itself is constructed using copies of the embedding S and an appropriate voltage graph. Full details are given in [9]. If the embeddings M and S are in orientable surfaces, then the resulting embedding of $K_{r(m-1)+1}$ is also orientable. As observed in [9], it is possible to use differently labelled embeddings R to bridge different white triangular faces W of $\bar{M}$. The result below was established in [9].

Theorem 3.1 *Suppose that* $m \equiv 3$ *or* $7 \pmod{12}$ *and that* $r \equiv 1$ *or* $3 \pmod 6$*. Suppose also that there are k differently labelled face 2-colourable triangular embeddings of $K_{r,r,r}$, all of which have a common parallel class of black triangular faces. Then we may construct $k^{(m-1)(m-3)/6}$ differently labelled face 2-colourable triangular embeddings of $K_{r(m-1)+1}$, all of which are orientable.*

The following result follows directly from this and Theorem 2.1.

Corollary 3.1.1 *Suppose that* $m \equiv 3$ *or* $7 \pmod{12}$, *that* $r = pq$ *where* $p = 3(2^{2t+1} - 1)$, $t \geq 1$ *and* q *is odd. Then there are at least*

$$\left(\frac{4p - 33}{3}\right)^{\left(\frac{p(p-3)}{16} - 1\right)\left(\frac{(m-1)(m-3)}{6}\right)}$$

differently labelled face 2-colourable triangular embeddings of $K_{r(m-1)+1}$, *each of which is orientable.*

Corollary 3.1.1 enables us to state the following result.

Corollary 3.1.2 *Suppose that* $n = 3q(2^{2t+1} - 1)(m - 1) + 1$, *where* $q > 1$ *is odd,* $t \geq 1$, *and* $m \equiv 3$ *or* $7 \pmod{12}$ *with* $m \geq 7$. *Put* $a = \dfrac{m-3}{96q^2(m-1)}$. *Then, as* $t \to \infty$, *there are at least* $n^{n^2(a - o(1))}$ *nonisomorphic face 2-colourable triangular embeddings of* K_n *in an orientable surface.*

Proof. In the estimate given by Corollary 3.1.1, write $r = \frac{n-1}{m-1}$. This gives the lower bound for the number of differently labelled face 2-colourable triangular embeddings of K_n as

$$\left[\frac{4(n-1)}{3q(m-1)} - 11\right]^{\left(a(n-1)^2\left(1 - \frac{3q(m-1)}{n-1}\right)\right)}$$

Dividing this expression by $n!$ (the maximum possible size of an isomorphism class), and making the usual estimates, gives the result. □

4 Concluding remarks

If, for example, we take $m = 7$ and $q = 3$ in Corollary 3.1.2, then we can deduce that for n of the form $54(2^{2t+1} - 1) + 1$, as $t \to \infty$, there are at least $n^{n^2\left(\frac{1}{1296} - o(1)\right)}$ nonisomorphic face 2-colourable triangular embeddings of K_n in an orientable surface. By taking m large and q to have its minimum value 3 in Corollary 3.1.2, our value for the constant a approaches $\frac{1}{864}$, but the resulting bound applies to a heavily restricted range of values of n. We conjecture that there exists a positive constant a such that for all $n \equiv 3$ or $7 \pmod{12}$ the number of nonisomorphic face 2-colourable triangular embeddings of K_n in an orientable surface is at least $n^{n^2(a - o(1))}$ as $n \to \infty$.

Our proof could be simplified and the final results improved if we had, in place of Lemma 2.2, for each integer $m = 2^s - 1$ a biembedding of the Latin square L formed by the Cayley table of the Steiner quasigroup corresponding to the projective STS(m). A further improvement would be obtained if for each n we had a biembedding of Latin squares of order n which had $O(n^3)$ 2-subsquares and a transversal. And yet another simplification and improvement could be made if, for each n satisfying the necessary conditions, we had a biembedding of STS(n)s which had $O(n^3)$ 2-subsquares.

We note that the bound obtained in [5] for the number of nonorientable face 2-colourable triangular embeddings of K_n can be improved for certain values of n to give essentially the same bound as in Corollary 3.1.2, since a slight generalization of Theorem 3.1 facilitates nonorientable embeddings. In fact, embeddings obtained in [7], as remarked at the end of that paper, already give this improved bound.

Computational results, such as [1] and [8] suggest that there are many more nonorientable face 2-colourable triangular embeddings of K_n than orientable ones. It may therefore seem surprising that we have essentially the same bound for orientable as for nonorientable triangular embeddings of K_n (though for a sparser class of values n). The explanation may well lie in the $o(1)$ error term.

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