On the number of triangular embeddings of complete graphs and complete tripartite graphs

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Abstract

We prove that for every prime number p and odd $m > 1$, as $s \to \infty$, there are at least $w^{w^2\left(\frac{1}{p^4 m^2} - o(1)\right)}$ face 2-colourable triangular embeddings of $K_{w,w,w}$, where $w = m \cdot p^s$. For both orientable and nonorientable embeddings, this result implies that for infinitely many infinite families of z, there is a constant $c > 0$ for which there are at least z^{cz^2} nonisomorphic face 2-colourable triangular embeddings of K_z.

1 Introduction

In proving Heawood's conjecture Ringel and Youngs ([17], [18]) constructed one triangular embedding of a complete graph K_z in a nonorientable surface for every $z \equiv 0,1 \pmod 3$, $z \geq 9$, and one triangular embedding of K_z in an orientable surface for every $z \equiv 0,3,4,7 \pmod{12}$, $z \geq 3$. Their constructions were based on current graphs. By varying the rotations at vertices of current graphs, Korzhik and Voss obtained $a \cdot 2^{bz}$ nonisomorphic triangular embeddings of these complete graphs (a and b being positive constants) in both the orientable and nonorientable cases, providing that z is sufficiently large, see [11], [12], [13], [14] and [15]. If a triangular embedding of K_z is face 2-colourable, then the two colour classes form two Steiner triple systems on z points. Face 2-colourability requires $z \equiv 1,3,7,9 \pmod{12}$ in the nonorientable case, and $z \equiv 3,7 \pmod{12}$ in the orientable case. In the papers [1], [6] (see also [2] for a survey) the authors constructed $a \cdot 2^{bz^2}$ nonisomorphic triangular embeddings of K_z (a and b being positive constants) in both the orientable and nonorientable cases, provided that z is sufficiently large and lies in certain congruence classes. For example, if $z \equiv 7,19 \pmod{36}$, then it is shown in [1] that, as $z \to \infty$, there are at least $2^{z^2(\frac{1}{54}-o(1))}$ face 2-colourable triangular embeddings of K_z in an orientable surface.

As mentioned in [8], an upper bound for the number of nonisomorphic triangular face 2-colourable embeddings of K_z is $z^{z^2/3}$. The following result is given in [3] (where the conditions $m > 1$ and $m \cdot 2^s \equiv 0,4 \pmod 6$ were unfortunately omitted).

Theorem 1.1 *Suppose that* $z = m \cdot 2^s(t-1)+1$ *where* $t \equiv 3,7 \pmod{12}$, $t \geq 7$, *m is odd,* $m > 1$, *and* $m \cdot 2^s \equiv 0,4 \pmod 6$. *Then as* $s \to \infty$, *there are at least*

$$z^{z^2\left(\frac{t-3}{192m^2(t-1)}-o(1)\right)}$$

nonisomorphic face 2-colourable triangular embeddings of K_z *in a nonorientable surface.*

In [5] it was explained how the constant $\frac{1}{192}$ may be improved by a factor of 2 to $\frac{1}{96}$.

As regards orientable embeddings, the following result is given in [8].

Theorem 1.2 *Suppose that* $z = 3m(2^{2s+1}-1)(t-1)+1$ *where* $t \equiv 3,7 \pmod{12}$, $t \geq 7$, *m is odd and* $m > 1$. *Then as* $s \to \infty$, *there are at least*

$$z^{z^2\left(\frac{t-3}{96m^2(t-1)}-o(1)\right)}$$

nonisomorphic face 2-colourable triangular embeddings of K_z *in an orientable surface.*

Hence, in both the orientable and nonorientable cases, the lower and upper bounds for the number of nonisomorphic face 2-colourable triangular embeddings of K_z are both of the form z^{bz^2} for a suitable constant $b > 0$, although the lower bound is only established for restricted infinite families of z. The primary purpose of this paper is to extend these results on the lower bound to a much wider range of values of z. We also obtain the improved version of Theorem 1.1 described in [5] and mentioned above.

As in [3], [6] and [8], a key ingredient is the construction of many differently labelled triangular embeddings of the complete regular tripartite graph $K_{w,w,w}$. Therefore, the next section is devoted to such graphs. Complete graphs are dealt with in the subsequent section. The reader is referred to [10] for terminology undefined in the paper.

2 Complete tripartite graphs

Suppose that ρ is a face 2-colourable triangular embedding of $K_{n,n,n}$. As shown in [4], such an embedding is necessarily in an orientable surface. If the tripartition of $K_{n,n,n}$ is taken to define the row, column and entry labels of an $n \times n$ array, then in each of the two colours, every prescribed pair of row and column labels is in a triangle with a unique entry label; every pair of row and entry labels is in a triangle with a unique column label; and every pair of column and entry labels is in a triangle with a unique row label. Hence, the triangles of each colour class determine a Latin square. Denote by R and R' the two Latin squares obtained from ρ in this way. Then we say that R *biembeds* with R' and we write $R \bowtie R'$. With a slight abuse of notation we also write $\rho = R \bowtie R'$. The triangles corresponding to R will be taken to be coloured white, while those corresponding to R' will be taken to be black.

Let R be a Latin square of order r. A *transversal* in R is a set of r distinct entry labels occurring in r distinct rows and r distinct columns. In an embedding $\rho = R \bowtie R'$, a transversal in R (R') corresponds to a *parallel class* of faces, i.e. to a set of r vertex disjoint triangles, coloured white (black).

In [6], see also [5], there is a product construction which creates a face 2-colourable triangular embedding of $K_{mn,mn,mn}$ from face 2-colourable triangular embeddings of $K_{n,n,n}$ and $K_{m,m,m}$. We now recall this construction.

Take m face 2-colourable triangular embeddings of $K_{n,n,n}$, $\varphi_t = L \bowtie L'_t$, $0 \leq t \leq m-1$, $m > 1$. Observe that these embeddings have identical sets of white triangles. To be able to distinguish the vertices of these triangular embeddings, suppose that the vertex set of φ_t is $R_t \cup C_t \cup E_t$, where $R_t = \{r^t_0, r^t_1, \ldots, r^t_{n-1}\}$, $C_t = \{c^t_0, c^t_1, \ldots, c^t_{n-1}\}$ and $E_t = \{e^t_0, e^t_1, \ldots, e^t_{n-1}\}$ are the three sets forming the tripartition of $K_{n,n,n}$. We assume that the vertices of R_t, C_t and E_t correspond to rows, columns and entries, respectively, of both L and L'_t. The triple (r^0_i, c^0_j, e^0_k) is a white triangle of φ_0 if and only if (r^t_i, c^t_j, e^t_k) is a white triangle of φ_t for any t, $0 \leq t \leq m-1$.

Now take n^2 face 2-colourable triangular embeddings of $K_{m,m,m}$, $\psi_{i,j} = Q_{i,j} \bowtie Q'_{i,j}$, $0 \leq i, j \leq n-1$, in which all the squares $Q'_{i,j}$ have a common transversal T, corresponding to a parallel class of black triangles in $\psi_{i,j}$. We suppose that the vertex set of $\psi_{i,j}$ is $R' \cup C' \cup V'$, where $R' = \{r'_0, r'_1, \ldots, r'_{m-1}\}$, $C' = \{c'_0, c'_1, \ldots, c'_{m-1}\}$ and $V' = \{e'_0, e'_1, \ldots, e'_{m-1}\}$ are the three sets forming the tripartition of $K_{m,m,m}$. We assume that the vertices of R', C' and V' correspond to rows, columns and entries, respectively, of both $Q_{i,j}$ and $Q'_{i,j}$. Suppose that the parallel class of black triangles, common to all $\psi_{i,j}$, is $T = \{(r'_{\alpha_t}, c'_{\beta_t}, e'_{\gamma_t}) : 0 \le t \le m-1\}$, where α, β and γ are permutations of $\{0, 1, \ldots, m-1\}$, and we can assume that α is the identity, i.e. $\alpha_t = t$ for $0 \le t \le m-1$. We do not need to distinguish the vertices of ψ_{i_1,j_1} from those of ψ_{i_2,j_2}, as in the final embedding of $K_{mn,mn,mn}$ we will use only the names of vertices appearing in φ_t, $0 \le t \le m-1$.

Consider one white triangle of φ_0, say (r_i^0, c_j^0, e_k^0), and cut out this triangle from φ_0 and its copies (r_i^t, c_j^t, e_k^t) from φ_t, $1 \le t \le m-1$. This results in m surfaces with m disjoint boundaries. Analogously, cut out the m black triangles $(r'_{\alpha_t}, c'_{\beta_t}, e'_{\gamma_t})$ of the parallel class T from the map $\psi_{i,j}$. Observe that the indices i and j of $\psi_{i,j}$ are the same as the indices of the vertices r_i^0 and c_j^0. Now glue the boundaries of (r_i^t, c_j^t, e_k^t) and $(r'_{\alpha_t}, c'_{\beta_t}, e'_{\gamma_t})$, $0 \le t \le m-1$, so that r_i^t, c_j^t and e_k^t are identified with r'_{α_t}, c'_{β_t} and e'_{γ_t}, respectively.

Repeat the procedure with each of the white triangles of φ_0 in turn. At each subsequent step after the first, cutting out the white triangles leaves a single surface but still with m disjoint boundaries. Denote the resulting embedding by ρ. The names of vertices of ρ are inherited from φ_t, $0 \le t \le m-1$, so that the resulting embedded graph is tripartite with tripartition $\cup_{t=0}^{m-1} R_t$, $\cup_{t=0}^{m-1} C_t$ and $\cup_{t=0}^{m-1} E_t$. Every edge $r_i^0 c_j^0$, $r_i^0 e_k^0$ or $c_j^0 e_k^0$ is in a unique white triangle in φ_0, say (r_i^0, c_j^0, e_k^0). Hence, for every s and t, $0 \le s, t \le m-1$ and $s \ne t$, the edge $r_i^s c_j^t$ ($r_i^s e_k^t$, $c_j^s e_k^t$) is added just once in the construction, namely when gluing $\psi_{i,j}$. Thus the underlying graph of the triangular embedding is the complete tripartite graph $K_{mn,mn,mn}$. Observe that the triangular embedding ρ is face 2-colourable because the holes in φ's come from white triangles while those in ψ's come from black triangles. Consequently, ρ triangulates an orientable surface.

In the simplest form, when all φ's are identical maps and also all ψ's are identical maps, the following result from [5] gives an easy description of the two Latin squares involved in ρ.

Theorem 2.1 *Suppose that $L \bowtie L'$, where L and L' are Latin squares of order n and have row, column and entry labels $\{0, 1, \ldots, n-1\}$. Suppose also that $Q \bowtie Q'$, where Q and Q' are Latin squares of order m and have row, column and entry labels $\{0, 1, \ldots, m-1\}$, and that the square Q' has a transversal T. Define $Q(L)$ and $Q'(L, T, L')$, Latin squares of order mn, with row, column and entry labels $\{0, 1, \ldots, mn-1\}$, so that for every u, v, i and j, $0 \le u, v \le m-1$ and $0 \le i, j, \le n-1$, we have*

$$\begin{aligned} Q(L)(nu+i, nv+j) &= nQ(u,v) + L(i,j), \\ Q'(L,T,L')(nu+i, nv+j) &= nQ'(u,v) + k, \end{aligned}$$

$$\text{where } k = \begin{cases} L(i,j) & \text{if } (r_u, c_v, e_w) \notin T \text{ for any } w, \\ L'(i,j) & \text{if there exists } w \text{ such that } (r_u, c_v, e_w) \in T. \end{cases}$$

Then $Q(L) \bowtie Q'(L,T,L')$.

In fact, $\rho = Q(L) \bowtie Q'(L,T,L')$, where ρ is the map described above. The square $Q(L)$ is partitioned into $n \times n$ subsquares which are just relabelled copies of L. The square $Q'(L,T,L')$ has a similar structure but the subsquares corresponding to the transversal T are relabelled copies of L'. Note that if L' has a transversal, then among the relabelled copies of L' one can find a transversal in $Q'(L,T,L')$. This feature facilitates re-application of the construction.

Observe that if Q and L are Cayley tables of groups $\mathcal{Q}$ and $\mathcal{L}$, represented respectively on $\{0,1,\dots,m-1\}$ and $\{0,1,\dots,n-1\}$, then $Q(L)$ is the Cayley table of the direct product $\mathcal{Q} \times \mathcal{L}$ represented on $\{0,1,\dots,mn-1\}$. Denote by C_r the Cayley table of a cyclic group $\mathcal{C}_r$. Then $C_r(C_r) = C_r^2$ is a Cayley table of $\mathcal{C}_r \times \mathcal{C}_r$. Repeating this process we define C_r^s, the Cayley table of $\mathcal{C}_r^s$. We then have the following result.

Theorem 2.2 *Suppose that* p *is prime and* $s \geq 1$, *where* $(p,s) \neq (2,1)$ *or* $(2,2)$. *Then there is a square* A_p^s *having a transversal and such that* $C_p^s \bowtie A_p^s$.

Proof. The case $p = 2$ is covered by Theorem 3.1 of [5]. So consider the case when p is odd. Observe that $C_p(i,j) = i + j \pmod p$. Denote by C_p^* the Latin square of order p, such that $C_p^*(i,j) = i + j - 1 \pmod p$. As shown in [4], $C_p \bowtie C_p^*$, the embedding being the unique regular triangular embedding of a complete tripartite graph, see [16]. Hence, if we take $A_p^1 = C_p^*$, we have $C_p^1 \bowtie A_p^1$. Moreover, as p is an odd number, the square $C_p^* = A_p^1$ has a transversal T_1 on the main diagonal, $T_1 = \{(i,i,2i-1) : 0 \leq i \leq p-1\}$, the arithmetic being modulo p. Consequently, $C_p(C_p) \bowtie A_p^1(C_p, T_1, A_p^1)$.

Define $A_p^2 = C_p^*(C_p, T, C_p^*)$ so that $C_p^2 \bowtie A_p^2$. Then A_p^2 has a transversal T_2, which appears in the relabelled copies of A_p^1 in A_p^2, as remarked above. In this case the transversal T_2 is again on the main diagonal. By continuing this process, we obtain the desired result. ■

We remark that in [7], Theorem 2.2 was generalized to give biembeddings of the Cayley tables of all Abelian groups except $\mathcal{C}_2^2$. However, in some cases the second square A lacks a transversal.

Let A be a Latin square or a subsquare of a Latin square. We say that (i,j,k) is a triangle of A if $k = A(i,j)$. If there are bijections from the row, column and entry labels of A to the row, column and entry labels (respectively) of C_p which map A to C_p, then A is said to be *isotopic* to C_p, and we describe A as a C_p*-square* or a C_p*-subsquare*.

Theorem 2.3 *Let* p *be prime and* $m \geq p$. *Then a Latin square of order* m *has at most* $\frac{m^2(m-1)}{p^2(p-1)}$ C_p*-subsquares. Further, let* $n = p^s$. *Then* C_p^s *has exactly* $\frac{n^2(n-1)}{p^2(p-1)}$ C_p*-subsquares.*

Proof. Suppose that L is a Latin square of order m with row, column and entry labels $\{0,1,\dots,m-1\}$. Let i_0, j_0, j_1 be such that $0 \leq i_0, j_0, j_1 \leq m-1$

and $j_0 \neq j_1$. We show that there is at most one C_p-subsquare of L containing row i_0 and columns j_0 and j_1. Define $k_0 = L(i_0, j_0)$. Then having i_{t-1} and k_{t-1} defined, i_t and k_t may be defined recursively for $1 \leq t \leq p-1$ by taking $k_t = L(i_{t-1}, j_1)$ and i_t to be the unique row such that $L(i_t, j_0) = k_t$. Of course, it can happen that either $\{i_0, i_1, \ldots, i_{p-1}\}$ or $\{k_0, k_1, \ldots, k_{p-1}\}$ is not a set of p different values. But as p is prime, if row i_0 and columns j_0 and j_1 are in a C_p-subsquare, then this subsquare must contain rows $i_0, i_1, \ldots, i_{p-1}$ and entries $k_0, k_1, \ldots, k_{p-1}$ and these rows (entries) are distinct.

Now define j_t to be the unique column such that $L(i_0, j_t) = k_t$, $2 \leq t \leq p-1$. If row i_0 and columns j_0 and j_1 are in a C_p-subsquare, then this subsquare must contain columns $j_0, j_1, \ldots, j_{p-1}$ and these columns have to be distinct. Moreover, we must have $L(i_a, j_b) = k_{a+b}$, the subscript arithmetic being modulo p with $0 \leq a, b, a+b \leq p-1$.

Hence, the row i_0 and columns j_0 and j_1 determine at most one C_p-subsquare. As a consequence, there are at most $m^2(m-1)$ C_p-subsquares in L with prescribed first row number and first and second column numbers. Since we do not distinguish the order of rows or columns in C_p-subsquares, the total number of C_p-subsquares in L is at most $\frac{m^2(m-1)}{p^2(p-1)}$.

To prove the second part of the theorem, it suffices to show that for each row i_0 and for each pair of distinct columns j_0 and j_1 there is a C_p-subsquare of C_p^s containing row i_0 and columns j_0 and j_1. We take the row, column and entry labels of C_p^s to be the elements of the commutative group $\mathcal{C}_p^s$ written additively so that the entry in row i column j is $i+j$. Define $z = j_1 - j_0$ and then define $i_t = i_0 + tz$, $j_t = j_0 + tz$ and $k_t = i_0 + j_0 + tz$ for $0 \leq t \leq p-1$. As p is prime, every element of $\mathcal{C}_p^s$, except the unit element 0, has order p. Since $j_0 \neq j_1$, we have $z \neq 0$. Consequently, $\{i_0, i_1, \ldots, i_{p-1}\}$, $\{j_0, j_1, \ldots, j_{p-1}\}$ and $\{k_0, k_1, \ldots, k_{p-1}\}$ are sets having exactly p elements. Let a and b be numbers such that $0 \leq a, b \leq p-1$. Then

$$C_p^s(i_a, j_b) = i_a + j_b = (i_0 + az) + (j_0 + bz) = i_0 + j_0 + (a+b)z = k_{a+b},$$

where $a+b$ is considered modulo p. In other words, the identified rows and columns determine a C_p-subsquare of C_p^s. ■

Now we change the gluing process in the product construction described earlier. Suppose that the Latin square L contains a C_p-subsquare. For notational convenience, assume that this cyclic subsquare contains the rows $0, 1, \ldots, p-1$, columns $0, 1, \ldots, p-1$ and entries $0, 1, \ldots, p-1$, so that $L(a, b) = a+b$, the arithmetic being modulo p with $0 \leq a, b, a+b \leq p-1$. Denote by P the set of p^2 white triangles corresponding to this subsquare of L in $\varphi_0 = L \bowtie L'_0$. We change the gluing process on the triangles of P. As previously, cut out all the white triangles of φ_t, $0 \leq t \leq m-1$, and cut out all the black triangles of the transversals T in all $\psi_{i,j}$, $0 \leq i, j \leq m-1$. If (r_i^0, c_j^0, e_k^0) is a white triangle of φ_0 which does not belong to P, then do the gluing as above. However, if $(r_a^0, c_b^0, e_{a+b}^0)$ is any one of the p^2 triangles of P, then take $\psi_{a,b}$ and for $0 \leq t < m-1$ identify $(r_a^t, c_b^t, e_{a+b}^t)$ with $(r'_{\alpha_t}, c'_{\beta_t}, e'_{\gamma_t})$, as earlier, while $(r'_{\alpha_{(m-1)}}, c'_{\beta_{(m-1)}}, e'_{\gamma_{(m-1)}})$ will be identified with $(r_a^{m-1}, c_{b+1}^{m-1}, e_{a+b+1}^{m-1})$, the subscript arithmetic being modulo p. Clearly, such an identification leads to face 2-colourable triangular embedding in a surface. The question is, what is the underlying graph? We show that it is $K_{mn,mn,mn}$.

Obviously, if (r_i^0, c_j^0, e_k^0) is a triangle which is not in P, then in the resulting embedding, exactly as previously, we have edges $r_i^s c_j^t$, $r_i^s e_k^t$, $c_j^s r_i^t$, $c_j^s e_k^t$, $e_k^s r_i^t$ and $e_k^s c_j^t$ for all s and t, $0 \le s, t \le m-1$. Even if (r_i^0, c_j^0, e_k^0) is a triangle of P, then in the resulting embedding we have edges $r_i^s c_j^t$, $r_i^s e_k^t$, $c_j^s r_i^t$, $c_j^s e_k^t$, $e_k^s r_i^t$ and $e_k^s c_j^t$ for all s and t, $0 \le s, t < m-1$. Hence, we have only to check these edges when $s < t = m-1$ and then only if (r_i^0, c_j^0, e_k^0) is a triangle of P. The following table describes how these edges are formed, where $0 \le x, y \le p-1$ and arithmetic is modulo p.

$r_x^s c_y^{m-1}$: $\psi_{x,y-1}$ connects $(r_x^s, c_{y-1}^s, e_{x+y-1}^s)$ with $(r_x^{m-1}, c_y^{m-1}, e_{x+y}^{m-1})$;
$r_x^s e_y^{m-1}$: $\psi_{x,y-x-1}$ connects $(r_x^s, c_{y-x-1}^s, e_{y-1}^s)$ with $(r_x^{m-1}, c_{y-x}^{m-1}, e_y^{m-1})$;
$c_x^s r_y^{m-1}$: $\psi_{y,x}$ connects $(r_y^s, c_x^s, e_{x+y}^s)$ with $(r_y^{m-1}, c_{x+1}^{m-1}, e_{x+y+1}^{m-1})$;
$c_x^s e_y^{m-1}$: $\psi_{y-x-1,x}$ connects $(r_{y-x-1}^s, c_x^s, e_{y-1}^s)$ with $(r_{y-x-1}^{m-1}, c_{x+1}^{m-1}, e_y^{m-1})$;
$e_x^s r_y^{m-1}$: $\psi_{y,x-y}$ connects $(r_y^s, c_{x-y}^s, e_x^s)$ with $(r_y^{m-1}, c_{x-y+1}^{m-1}, e_{x+1}^{m-1})$;
$e_x^s c_y^{m-1}$: $\psi_{x-y+1,y-1}$ connects $(r_{x-y+1}^s, c_{y-1}^s, e_x^s)$ with $(r_{x-y+1}^{m-1}, c_y^{m-1}, e_{x+1}^{m-1})$.

Hence, every edge of $K_{mn,mn,mn}$ appears at least once. It is obvious that no edge can appear more than once due to the fact that the original gluing procedure led to an embedding of $K_{mn,mn,mn}$ and the number of edges created by the new gluing process is the same as that for the original. Let us call this gluing on triangles of P *nonstandard*, while the gluing mentioned before Theorem 2.1 we will call *standard*.

Next we show that nonstandard gluings lead to differently labelled final maps. Using the notation established earlier, we take $m > 1$ face 2-colourable triangular embeddings $\varphi_0, \varphi_1, \dots, \varphi_{m-1}$ of $K_{n,n,n}$, such that all φ_t, $0 \le t \le m-1$, have identically labelled white triangles (up to the upper index), and we suppose that we have n^2 face 2-colourable triangular embeddings $\psi_{i,j}$, $0 \le i, j \le n-1$, all of which have an identically labelled parallel class T of black triangles. Then the following result may be obtained.

Theorem 2.4 *Suppose that there is a cyclic subsquare P of prime order p in the white square of φ_0 and that (r_i^0, c_j^0, e_k^0) is a triangle of P. Take ρ_1 to be the map obtained from the construction by applying nonstandard gluing to P. Take ρ_2 to be the map obtained either by*

(a) applying standard gluing to P, or

(b) taking a different cyclic subsquare P^* ($\neq P$) of order p in the white square of φ_0 with (r_i^0, c_j^0, e_k^0) a triangle of P^*, and applying nonstandard gluing to P^*.

Then ρ_1 and ρ_2 are differently labelled face 2-colourable triangular embeddings of $K_{mn,mn,mn}$. Moreover, if each φ_t has a parallel class of black triangles, $0 \leq t \leq m-1$, then both ρ_1 and ρ_2 have an identically labelled parallel class of black triangles.

Proof. We use the labelling of vertices of φ_t and $\psi_{i,j}$ described above, taking $T = \{(r'_{\alpha_t}, c'_{\beta_t}, e'_{\gamma_t}) : 0 \leq t \leq m-1\}$. For notational convenience, assume that P contains the triangles (r_i^0, c_j^0, e_k^0), where $k = i+j \pmod p$ and $0 \leq i, j, k \leq p-1$.

Consider the map $\psi_{i,j}$ where $0 \leq i, j \leq p-1$. In $\psi_{i,j}$ there is a black triangle $(r'_{\alpha_{(m-1)}}, c'_{\beta_{(m-1)}}, e'_{\gamma_{(m-1)}})$, and this triangle is adjacent to a white triangle $(r'_{\alpha_{(m-1)}}, c'_{\beta_{(m-1)}}, e'_{\gamma_q})$ for some q, $0 \leq q < m-1$. Note that $q \neq m-1$.

If we glue P in a standard way, then in the gluing of $\psi_{i,j}$ we identify the borders of $(r_i^{m-1}, c_j^{m-1}, e_k^{m-1})$ and $(r'_{\alpha_{(m-1)}}, c'_{\beta_{(m-1)}}, e'_{\gamma_{(m-1)}})$, and we also identify the borders of (r_i^q, c_j^q, e_k^q) and $(r'_{\alpha_q}, c'_{\beta_q}, e'_{\gamma_q})$. Hence if ρ_2 is obtained by standard gluing, then this process yields a white triangle $(r_i^{m-1}, c_j^{m-1}, e_k^q)$ in the resulting map (in which vertices inherit the names of vertices of φ_t, $0 \leq t \leq m-1$).

Now consider the nonstandard method of gluing on P. In the gluing of $\psi_{i,j}$ we identify the borders of (r_i^q, c_j^q, e_k^q) and $(r'_{\alpha_q}, c'_{\beta_q}, e'_{\gamma_q})$, and we also identify the borders of $(r_i^{m-1}, c_{j+1}^{m-1}, e_{k+1}^{m-1})$ and $(r'_{\alpha_{(m-1)}}, c'_{\beta_{(m-1)}}, e'_{\gamma_{(m-1)}})$, where $j+1$ and $k+1$ are considered modulo p. This process yields a white triangle $(r_i^{m-1}, c_{j+1}^{m-1}, e_k^q)$ in the resulting map ρ_1. Hence in case (a) ρ_1 has a different white triangle from ρ_2.

Next suppose that case (b) applies, that is to say that ρ_2 is obtained by applying nonstandard gluing to P^* ($\neq P$) when both P and P^* contain the triangle (r_i^0, c_j^0, e_k^0). By the proof of Theorem 2.3, there cannot be a second c-vertex c_ℓ^0 ($\ell \neq j$) common to both P and P^*, since in such a case we would have $P = P^*$. Therefore, the unique c-vertex common to both P and P^* is c_j^0. Thus in the nonstandard gluing of $\psi_{i,j}$ we identify the borders of (r_i^q, c_j^q, e_k^q) and $(r'_{\alpha_q}, c'_{\beta_q}, e'_{\gamma_q})$ as earlier, and we also identify the borders of $(r_i^{m-1}, c_{j^*}^{m-1}, e_{k^*}^{m-1})$ and $(r'_{\alpha_{(m-1)}}, c'_{\beta_{(m-1)}}, e'_{\gamma_{(m-1)}})$ for some j^* and k^*, where $c_{j^*}^0$ is a vertex which does not appear in P. This process yields a white triangle $(r_i^{m-1}, c_{j^*}^{m-1}, e_k^q)$ in the resulting map ρ_2. As a consequence in case (b), the maps ρ_1 and ρ_2 have differently labelled white triangles containing the edge $r_i^{m-1} e_k^q$.

In both cases (a) and (b), note that ρ_1 and ρ_2 are not equivalent under exchange of colours, since any black triangle containing the edge $r_i^{m-1}c_\ell^{m-1}$, $0 \le \ell \le n-1$, has the form $(r_i^{m-1}, c_\ell^{m-1}, e_h^{m-1})$ for some h.

As described above, both ρ_1 and ρ_2 are face 2-colourable triangular embeddings of $K_{mn,mn,mn}$. Suppose that T_t is a parallel class of black triangles in φ_t, $0 \le t \le m-1$. Since we neither cut out nor relabelled the black triangles of φ_t, the system $\cup_{t=0}^{m-1} T_t$ is a parallel class of identically labelled black triangles in both ρ_1 and ρ_2. ■

The previous theorem explains how to construct differently labelled face 2-colourable triangular embeddings of $K_{mn,mn,mn}$. We will show that a large number of such embeddings may be produced by this technique. We say that a set S of C_p-subsquares of a Latin square L is an *independent set of C_p-subsquares* if no two elements of S share a common triangle. (Observe that this independence relates to triangles and not to row, column or entry labels.)

Theorem 2.5 *Suppose that in a Latin square L of order n there are r different C_p-subsquares. Then the number of distinct independent sets of C_p-subsquares in L is at least*

$$\left(p^2\frac{n-1}{p-1}+1\right)^{\left(\frac{r(p-1)}{p^2(n-1)}-1\right)}.$$

Proof. Consider a triangle (r_i, c_j, e_k) of L. As shown in the proof of Theorem 2.3, for every $j' \neq j$ there is at most one C_p-subsquare of L containing (r_i, c_j, e_k) and a triple $(r_i, c_{j'}, e_{k'})$. Moreover, in such a case $p-1$ values of j' determine the same C_p-subsquare of L. Hence, the triangle (r_i, c_j, e_k) occurs in at most $\frac{n-1}{p-1}$ C_p-subsquares of L.

Let P be the set of triangles of a C_p-subsquare of L. Since P has p^2 triangles, a triangle of P occurs in at most $p^2\frac{n-1}{p-1}$ C_p-subsquares of L.

Denote by I_q the number of independent sets of C_p-subsquares containing exactly q C_p-subsquares. Then we have

$$I_q \ge \frac{r\Big(r-p^2\frac{n-1}{p-1}\Big)\Big(r-2p^2\frac{n-1}{p-1}\Big)\dots\Big(r-(q-1)p^2\frac{n-1}{p-1}\Big)}{q!}.$$

Put $Q = \left\lfloor \frac{r(p-1)}{p^2(n-1)} \right\rfloor$. Then from the above we deduce

$$I_q \ge \left(p^2\frac{n-1}{p-1}\right)^q \cdot \frac{Q(Q-1)(Q-2)\dots\Big(Q-(q-1)\Big)}{q!} = \left(p^2\frac{n-1}{p-1}\right)^q \binom{Q}{q}.$$

Of course, if $q > Q$ then we have just the trivial bound $I_q \ge 0$. Now summing I_q for all $q \le Q$ gives the bound on the number of independent sets of C_p-subsquares

$$\sum_{q=0}^{Q} I_q \ge \sum_{q=0}^{Q}\left(p^2\frac{n-1}{p-1}\right)^q\binom{Q}{q} = \left(1+p^2\frac{n-1}{p-1}\right)^Q \ge \left(p^2\frac{n-1}{p-1}+1\right)^{\left(\frac{r(p-1)}{p^2(n-1)}-1\right)}.$$

■

Now we combine our theorems.

Theorem 2.6 *Suppose that $n = p^s$ where p is prime, and if $p = 2$ assume that $s \geq 3$. Suppose also that $m > 1$ and that there is a face 2-colourable triangular embedding of $K_{m,m,m}$ having a parallel class of black triangles. Then there are at least*

$$\left(p^2 \frac{n-1}{p-1} + 1\right)^{\left(\frac{n^2}{p^4} - 1\right)}$$

differently labelled face 2-colourable triangular embeddings of $K_{mn,mn,mn}$, all of which have a common parallel class of black triangles. Furthermore, there are at least

$$\frac{\left(p^2 \frac{n-1}{p-1} + 1\right)^{\left(\frac{n^2}{p^4} - 1\right)}}{6\big((mn)!\big)^3}$$

nonisomorphic face 2-colourable triangular embeddings of $K_{mn,mn,mn}$.

Proof. We use the product construction, taking all φ_t, $0 \leq t \leq m-1$, to be isomorphic to the embedding $C_p^s \bowtie A_p^s$ guaranteed by Theorem 2.2, and all $\psi_{i,j}$, $0 \leq i, j \leq n-1$, to be isomorphic to the embedding of $K_{m,m,m}$ mentioned in the statement. Glue all the triangles in a standard way, except the triangles of one independent set of C_p-subsquares of C_p^s, which are glued in a nonstandard way. As proved in Theorem 2.4, for different independent sets of C_p-squares we obtain differently labelled face 2-colourable triangular embeddings of $K_{mn,mn,mn}$. By Theorem 2.3, there are $r = \frac{n^2(n-1)}{p^2(p-1)}$ different C_p-subsquares in C_p^s, and by Theorem 2.5, there are at least

$$\left(p^2 \frac{n-1}{p-1} + 1\right)^{\left(\frac{n^2}{p^4} - 1\right)}$$

different independent sets of C_p-subsquares in C_p^s, so that by Theorem 2.4, there are at least

$$\left(p^2 \frac{n-1}{p-1} + 1\right)^{\left(\frac{n^2}{p^4} - 1\right)}$$

differently labelled face 2-colourable triangular embeddings of $K_{mn,mn,mn}$. By Theorem 2.2, A_p^s has a transversal. As a consequence, all these differently labelled triangular embeddings of $K_{mn,mn,mn}$ have a common parallel class of black triangles.

The second part follows from the fact that the maximum possible size of an isomorphism class is $6((mn)!)^3$ in any embedding of $K_{mn,mn,mn}$. ■

We remark that the lower bound in the above theorem may easily be increased. First, observe that in the nonstandard gluing of triangles of P it is not necessary to identify $(r'_{\alpha_{(m-1)}}, c'_{\beta_{(m-1)}}, e'_{\gamma_{(m-1)}})$ with $(r_a^{m-1}, c_{b+1}^{m-1}, e_{a+b+1}^{m-1})$. Since p is prime, $(r'_{\alpha_{(m-1)}}, c'_{\beta_{(m-1)}}, e'_{\gamma_{(m-1)}})$ may be identified with $(r_a^{m-1}, c_{b+\ell}^{m-1}, e_{a+b+\ell}^{m-1})$ for any ℓ, $1 \leq \ell \leq p-1$. Also, the nonstandard gluing may be done not only at the $(m-1)$-th level; it can be done on any subset of levels $\{1, 2, \ldots, m-1\}$. The difficulty comes in showing that all the resulting embeddings are differently labelled. Although the bound might be increased, it appears that the exponent remains unaltered and consequently we will only consider the bound as stated in Theorem 2.6.

Corollary 2.6.1 *Suppose that $w = m \cdot p^s$, where p is prime, $m > 1$, and there exists a face 2-colourable triangular embedding of $K_{m,m,m}$ having a parallel class of black triangles. Then for $s \to \infty$ there are at least*

$$w^{w^2\left(\frac{1}{p^4 m^2} - o(1)\right)}$$

nonisomorphic face 2-colourable triangular embeddings of $K_{w,w,w}$, all of which have identically labelled parallel class of black triangles.

Proof. Let $n = p^s$. By Theorem 2.6, there are at least

$$\frac{\left(p^2 \frac{n-1}{p-1} + 1\right)^{\left(\frac{n^2}{p^4} - 1\right)}}{6\big((mn)!\big)^3}$$

nonisomorphic triangular embeddings satisfying the assumptions. Since $p^2 \frac{n-1}{p-1} + 1 \geq n = \frac{w}{m}$, the number of these triangular embeddings is at least

$$\frac{\left(\frac{w}{m}\right)^{\left(\frac{w^2}{p^4 m^2} - 1\right)}}{6\big(w!\big)^3} = w^{w^2\left(\frac{1}{p^4 m^2} - o(1)\right)}. \qquad \blacksquare$$

Observe that $C_m \bowtie C_m^*$ contains a parallel class T_1 of black triangles if m is odd (see the proof of Theorem 2.2 for the definition of C_m^* and T_1). Therefore, for $w = m \cdot p^s$, $m > 1$ being odd and p being prime, the assumptions of Corollary 2.6.1 are satisfied. The best bound in this corollary is obtained when $m = 3$.

3 Complete graphs

We now turn our attention to face 2-colourable triangular embeddings of complete graphs. The following result is an easy extension of Construction 5 of [6] and Theorem 3.1 of [3]. There are just two additional aspects to the proof. The first is that the capping operation described in the earlier papers may always be completed if at least one of w and $\frac{t-1}{2}$ is odd, and the second is that if any of the ingredients are nonorientable embeddings then so is the final resulting embedding.

Theorem 3.1 *Suppose that $t \equiv 1, 3, 7, 9 \pmod{12}$ and $w \equiv 0, 1, 3, 4 \pmod{6}$, where at least one of w and $\frac{t-1}{2}$ is odd. Moreover, suppose that there is a face 2-colourable triangular embedding of K_t, a face 2-colourable triangular embedding of K_{2w+1}, and that there are r differently labelled face 2-colourable triangular embeddings of $K_{w,w,w}$, all having an identically labelled parallel class of black triangles. Then there are at least*

$$r^{\frac{(t-1)(t-3)}{6}}$$

differently labelled face 2-colourable triangular embeddings of $K_{w(t-1)+1}$. If either the embedding of K_t or that of K_{2w+1} is nonorientable, then the embeddings of $K_{w(t-1)+1}$ are nonorientable. If the embeddings of both K_t and K_{2w+1} are orientable, then the embeddings of $K_{w(t-1)+1}$ are orientable.

It was shown in [17] (see also [2]) that for $n \equiv 3, 9 \pmod{12}$ $(n > 3)$, and in [9] that for $n \equiv 1, 7 \pmod{12}$ $(n > 7)$, there exists a face 2-colourable triangular embedding of K_n in a nonorientable surface. Moreover, the statement $2w+1 \equiv 1, 3, 7, 9 \pmod{12}$ is equivalent to $w \equiv 0, 1, 3, 4 \pmod{6}$. Thus, for nonorientable triangular embeddings we have the following corollary of Theorems 2.6 and 3.1.

Corollary 3.1.1 *Suppose that $z = m \cdot p^s(t-1)+1$ where $t \equiv 1, 3, 7, 9 \pmod{12}$, p is prime, $m > 1$ is odd, $m \cdot p^s \equiv 0, 1, 3, 4 \pmod{6}$, and at least one of $m \cdot p^s$ and $\frac{t-1}{2}$ is odd. Then as $s \to \infty$, there are at least*

$$z^{z^2\left(\frac{t-3}{6p^4m^2(t-1)} - o(1)\right)}$$

nonisomorphic face 2-colourable triangular embeddings of K_z in a nonorientable surface.

We remark that weaker versions of this corollary corresponding to the special case $p = 2$ were given in [3] and [5]. The best bound in the corollary is obtained for $p = 2, m = 3$. These values give the following result.

Corollary 3.1.2 *Suppose that $z = 3 \cdot 2^s(t-1)+1$ where $t \equiv 3, 7 \pmod{12}$. Then as $s \to \infty$, there are at least*

$$z^{z^2\left(\frac{t-3}{864(t-1)} - o(1)\right)}$$

nonisomorphic face 2-colourable triangular embeddings of K_z in a nonorientable surface.

As a consequence of this, if $t \equiv 3, 7 \pmod{12}$ is sufficiently large and $z = 3 \cdot 2^s(t-1)+1$, then as $s \to \infty$ there are at least $z^{z^2/865}$ nonisomorphic face 2-colourable triangular embeddings of K_z in a nonorientable surface.

Next we consider orientable embeddings. It was shown by Ringel [17] that for $n \equiv 3 \pmod{12}$, and by Youngs [18] that for $n \equiv 7 \pmod{12}$, there exists a face 2-colourable triangular embedding of K_n in an orientable surface. The statement $2w+1 \equiv 3, 7 \pmod{12}$ is equivalent to $w \equiv 1, 3 \pmod 6$. Hence, we have the following corollary of Theorems 2.6 and 3.1.

Corollary 3.1.3 *Suppose that $z = m \cdot p^s(t-1)+1$ where $t \equiv 3, 7 \pmod{12}$, p is prime, $m > 1$ is odd, and $m \cdot p^s \equiv 1, 3 \pmod 6$. Then as $s \to \infty$, there are at least*

$$z^{z^2\left(\frac{t-3}{6p^4m^2(t-1)}-o(1)\right)}$$

nonisomorphic face 2-colourable triangular embeddings of K_z in an orientable surface.

Since the conditions of the corollary require that $p \geq 3$, this bound is not as good as that of Theorem 1.2. However, it is also of the form $z^{z^2(a-o(1))}$ and it also covers many more infinite families of z.

The best bound in the preceding corollary is obtained for $p = m = 3$. These values give the following result.

Corollary 3.1.4 *Suppose that $z = 3^{s+1}(t-1)+1$ where $t \equiv 3, 7 \pmod{12}$. Then as $s \to \infty$, there are at least*

$$z^{z^2\left(\frac{t-3}{4374(t-1)}-o(1)\right)}$$

nonisomorphic face 2-colourable triangular embeddings of K_z in an orientable surface.

As a consequence of this, if $t \equiv 3, 7 \pmod{12}$ is sufficiently large and $z = 3^{s+1}(t-1)+1$, then as $s \to \infty$, there are at least $z^{z^2/4375}$ nonisomorphic face 2-colourable triangular embeddings of K_z in an orientable surface.

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