Minimal non-selfcentric radially-maximal graphs of radius 4

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Abstract

There is a hypothesis that a non-selfcentric radially-maximal graph of radius r has at least $3r-1$ vertices. Using some recent results we prove this hypothesis for $r=4$.

1 Introduction and result

Let G be a graph. By $E(G)$ we denote the edge set of G, and by $\overline{G}$ we denote the complement of G. The radius of G is denoted by $r(G)$ and the diameter of G is denoted by $d(G)$. We say that the graph G is **radially-maximal** if $r(G+e) < r(G)$ for every edge $e \in E(\overline{G})$.

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Obviously, for every r there is a radially-maximal graph of radius r, as can be shown by complete graphs (in the case $r = 1$) and even cycles (in the case $r > 1$). However, both complete graphs and cycles are selfcentric graphs. Here we recall that a graph G is **selfcentric** if $r(G) = d(G)$. If $r(G) < d(G)$ then G is a **non-selfcentric** graph. One may expect that a graph is radially-maximal if it is either a very dense graph or a balanced (highly symmetric) one. Therefore, it is interesting that for $r \geq 3$ there are non-selfcentric radially-maximal graphs of radius r which are planar (such graphs are neither symmetric nor dense). In fact, in [1] we have the following conjecture:

Conjecture A. *Let G be a non-selfcentric radially-maximal graph with radius $r \geq 3$ on the minimum number of vertices. Then*

(a) *G has exactly $3r - 1$ vertices;*

(b) *G is planar;*

(c) *the maximum degree of G is 3 and the minimum degree of G is 1.*

Conjecture A was verified for the case $r = 3$. By an exhaustive computer search it was shown that there are just two non-selfcentric radially-maximal graphs of radius 3 on at most 8 vertices. These graphs are depicted in Figure 1. As one can see, they are planar, their minimum degree is 1, the maximum degree is 3, and each of them has exactly 8 vertices.

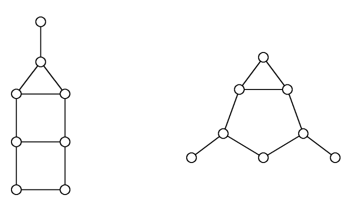

Figure 1

For higher values of r the conjecture was open, although by an extensive computer search we found that there are exactly 8 graphs of radius 4 fulfilling all the conclusions of Conjecture A. These graphs are depicted in Figure 2.

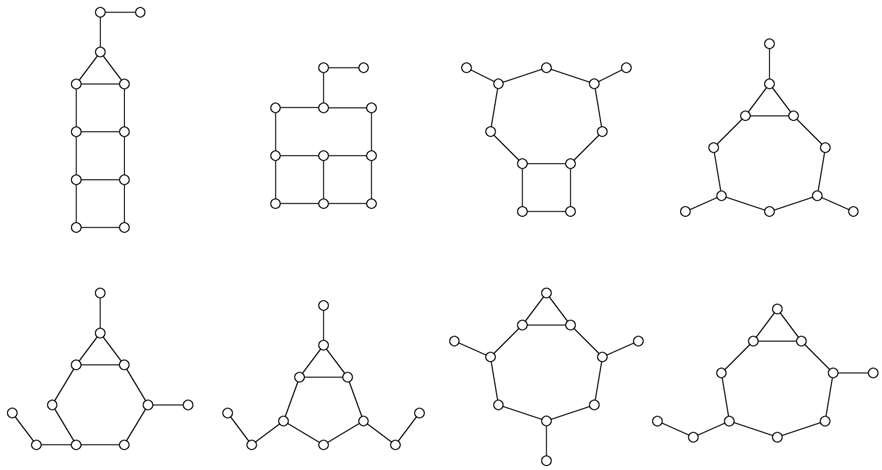

Figure 2

Although we are not able to prove the (a) part of Conjecture A in general, we have:

Assertion 1. *For every $r \geq 3$ there exists a non-selfcentric radially-maximal graph with radius r on $3r-1$ vertices.*

Hence, the (a) part of Conjecture A will be true if we prove that there are no non-selfcentric radially-maximal graphs with radius r on less than $3r-1$ vertices.

Let C be a cycle in a graph G. We say that C is a **geodesic cycle**, if for any two vertices of C, their distance on C equals their distance in G. Recently, in [2] Haviar, Hrnčiar and Monoszová proved the following beautiful statement:

Theorem B. *Let G be a graph with $r(G)=r$, $d(G) \leq 2r-2$, on at most $3r-2$ vertices. Then G contains a geodesic cycle of length either $2r$ or $2r+1$.*

Using this statement we are able to prove the (a) part of Conjecture A for $r=4$:

Theorem 2. *Let G be a non-selfcentric radially-maximal graph with radius 4 on the minimum number of vertices. Then G has exactly 11 vertices.*

The proofs of Assertion 1 and Theorem 2 are presented in the next section.

2 Proofs

Proof of Assertion 1. Let G_r be a graph obtained from the first graph in Figure 1 by extending the path on the top by $r-3$ vertices, and by extending the ladder at the bottom by $r-3$ new spokes. Then G_3 is the first graph in Figure 1, G_4 is the first graph in Figure 2, while a general version of G_r (in horizontal position) is depicted in Figure 3. The central vertices of G_r are denoted by c_1, c_2, c_3, c_4 and c_5, the vertices of the path at the top are $v_3, v_4, \ldots, v_r$, the vertices of one "leg" of the ladder are $w_3, w_4, \ldots, w_r$, and the vertices of the other "leg" are $z_3, z_4, \ldots, z_r$, see Figure 3.

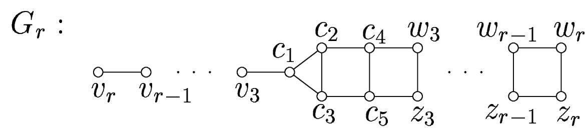

Figure 3

In the following, let us denote by $N_t^G(v)$ the set of vertices of G, which are at distance t from v, while the eccentricity of a vertex v is denoted by $e_G(v)$. Observe that $N_3^{G_3}(c_1) = \{w_3, z_3\}$, so that $N_r^{G_r}(c_1) = \{w_r, z_r\}$. Analogously, $N_r^{G_r}(c_2) = \{z_r\}$, $N_r^{G_r}(c_3) = \{w_r\}$, $N_r^{G_r}(c_4) = \{v_r\}$ and $N_r^{G_r}(c_5) = \{v_r\}$. Since the distance from v_r to w_r is $2r-2$, G_r is a non-selfcentric graph with radius r on $3r-1$ vertices. We show that G_r is radially-maximal.

First consider adding an edge c_ic_j to G_r. Since the distance from c_4 to v_r is less than r in $G_r + c_1c_4$, we have $e_{G_r+c_1c_4}(c_4) < r$. Analogously, $e_{G_r+c_1c_5}(c_5) < r$, $e_{G_r+c_2c_5}(c_2) < r$ and $e_{G_r+c_3c_4}(c_3) < r$.

Further, $e_{G_r+c_iv_j}(c_4) < r$ if $i \in \{1,2,4\}$ and $e_{G_r+c_iv_j}(c_5) < r$ if $i \in \{3,5\}$. Moreover, $e_{G_r+c_iw_j}(c_3) < r$ and $e_{G_r+c_iz_j}(c_2) < r$.

It remains to check adding an edge joining two non-central vertices. Obviously, $e_{G_r+v_iv_j}(c_4) < r$, $e_{G_r+w_iw_j}(c_3) < r$ and $e_{G_r+z_iz_j}(c_2) < r$. If $i < j$ then $e_{G_r+w_iz_j}(c_2) < r$ and if $i > j$ then $e_{G_r+w_iz_j}(c_3) < r$. Finally, if $i \geq j$ then $e_{G_r+v_iw_j}(c_4) < r$ and if $i < j$ then $e_{G_r+v_iw_j}(c_1) < r$. By symmetry, the case of adding v_iz_j can be reduced to the previous one. $\square$

For two vertices, say u and v, by $d_G(u,v)$ we denote their distance in G. In the proof of Theorem 2 we use the following statement:

Lemma 3. *Let G be a radially-maximal graph of radius r and diameter d. Then $d \le 2r-2$.*

Proof. Let $P = (v_0, v_1, \ldots, v_d)$ be a path on which the diameter of G is achieved. Denote by G' the graph obtained from G by adding the edge v_0v_d, and denote by P' the cycle $(v_0, v_1, \ldots, v_d, v_0)$ of G'.

Since G is a radially-maximal graph, we have $r(G') \le r-1$. Let v be a vertex of G' such that $e_{G'}(v) = r(G') \le r-1$. As $e_G(v) \ge r$, it holds $d_G(v,v_0) \ne d_G(v,v_d)$. Without loss of generality we may assume that $d_G(v,v_0) < d_G(v,v_d)$. Then $d_{G'}(v,v_0) = d_{G'}(v,v_d) - 1$. Denote $d_{v_0} = d_{G'}(v,v_0) = d_G(v,v_0)$. Further, denote by P_i (P'_i) the set of vertices of P (P') which distance to v in G (G') is at most i.

If $i \le d_{v_0}$ then $|P_i| \le 2i+1$, since otherwise P would not be a diametrical path. (There would be a short-cut via v.) As $i \le d_{v_0}$, $|P'_i| \le 2i+1$ as well.

If $i > d_{v_0}$ then $|P_i| \le 2d_{v_0} + 1 + (i - d_{v_0})$, since otherwise P would not be a diametrical path. (Observe that $d_G(v,v_0) = d_{v_0}$.) However, the presence of the edge v_0v_d in G' causes that P'_i contains also some vertices which are not in P_i. Since P is a diametrical path, there are only $i - d_{v_0}$ vertices in $P'_i - P_i$, namely $v_d, v_{d-1}, \ldots, v_{d-(i-d_{v_0})+1}$. Hence, $|P'_i| = |P_i| + (i - d_{v_0}) \le 2d_{v_0} + 1 + (i - d_{v_0}) + (i - d_{v_0}) = 2i+1$.

Thus, for every i we have $|P'_i| \le 2i+1$. Since $e_{G'}(v) \le r-1$, the set P'_{r-1} contains all the vertices of P'. Hence, $|P'| = |P| \le 2(r-1)+1 = 2r-1$, so that the length of P is at most $2r-2$. □

Every graph of radius r can be completed into a radially-maximal graph of radius r by adding edges which do not decrease the radius. In the following proof, the edges $e \in E(\overline{G})$ such that $r(G) = r(G+e)$ are called r-**admissible**. However, we may not add any admissible edge to G, since we like to remain in the class of non-selfcentric graphs. Hence, the edges $e \in E(\overline{G})$ such that $G+e$ is not selfcentric, are called s-**admissible**. Finally, we like to preserve a special substructure in G. There will be a cycle C in G which has to be geodesic. So that the edges $e \in E(\overline{G})$ for which C remains a geodesic cycle in $G+e$, are called g-**admissible**. We may join these notions. For example, if an edge $e \in E(\overline{G})$ is both s-admissible and g-admissible, then we say that e is sg-admissible. Finally, an edge is **admissible** if it is rsg-admissible.

Observe that if there are two edges, say e and f, which are admissible in G, then f is not necessarily an admissible edge in $G+e$. Hence, there may be more radially-maximal graphs obtained by adding edges to G.

In the following proof we use the notation introduced above. However, if we examine a specific edge, say h, and we realize that our graph cannot contain this edge, then in the sequel h is not an a-admissible edge for any $a \in \{r, s, g\}$.

Proof of Theorem 2. Suppose that there is a non-selfcentric radially-maximal graph with radius $r = 4$ on at most $3r - 2 = 10$ vertices. As there is no radially-maximal graph of radius r with diameter $d > 2r - 2$ by Lemma 3, our graph contains a geodesic cycle of length either 9 or 8, by Theorem B.

First suppose that a non-selfcentric radially-maximal graph with radius 4 on at most 10 vertices contains a geodesic cycle $(v_0, v_1, \ldots, v_8, v_0)$ of length 9. Since the cycle C is a selfcentric graph, there must be another vertex, say v_9, adjacent to a vertex of C. Without loss of generality we may assume that v_2v_9 is an edge in our graph. Denote $G = C + v_2v_9$. Obviously, G has 10 vertices.

Observe that G is not a radially-maximal graph. There are four rg-admissible edges in G, namely v_0v_9, v_1v_9, v_3v_9 and v_4v_9. But the edges v_0v_9 and v_4v_9 are not s-admissible. Since the graphs $G+v_1v_9$ and $G+v_3v_9$ are isomorphic, without loss of generality we may complete the graph G by adding the edge v_1v_9. Denote $G' = G+v_1v_9$. But G' is not a radially-maximal graph, as the edge v_3v_9 is r-admissible in G'. However, there are no sg-admissible edges in G', which contradicts our assumption that a non-selfcentric radially-maximal graph with radius 4 on at most 10 vertices contains a geodesic cycle of length 9.

Thus, suppose that there is a non-selfcentric radially-maximal graph with radius 4 on at most 10 vertices which contains a geodesic cycle $(v_0, v_1, \ldots, v_7, v_0)$ of length 8. Since a cycle is a selfcentric graph, there must be another vertex, say v_8, adjacent to C by an edge, say v_2v_8. Observe that v_1v_8 is an r-admissible edge in $C + v_2v_8$, so that $C + v_2v_8$ is not a radially-maximal graph. But there are no sg-admissible edges in $C + v_2v_8$. Thus, there must be another vertex, say v_9. Since now we attained the upper bound for the number of vertices, there are no other vertices in our graph. In what follows we distinguish two cases with several subcases each.

Case 1: Our graph contains the edge v_8v_9. Denote $G = C + v_2v_8 + v_8v_9$. As v_1v_8 is an r-admissible edge in G, the graph G is not radially-maximal.

However, there are only seven rsg-admissible edges in G, namely v_0v_8, v_1v_8, v_3v_8, v_4v_8, v_1v_9, v_2v_9 and v_3v_9. In fact, there are no other sg-admissible edges in G.

First consider the graph $G' = G+v_1v_9$. Since the edge v_2v_9 is r-admissible in G', this graph is not radially-maximal. But there are only four admissible edges in G', namely v_1v_8, v_2v_9, v_0v_8 and v_3v_9. However, the unique admissible edge in $G' + v_1v_8$ is v_0v_8. Since v_2v_9 is r-admissible in $G' + v_1v_8 + v_0v_8$, adding the edge v_1v_8 to G' will not create the required radially-maximal graph. Hence, we can exclude the edge v_1v_8, and by symmetry also the edge v_2v_9, from the list of admissible edges for G'. But then there is no admissible edge in $G' + v_0v_8$ although $r(G' + v_0v_8 + v_1v_8) = 4$. Hence, also the edges v_0v_8 and v_3v_9 can be excluded from the list of admissible edges for G'. This shows that G' cannot be completed to a required radially-maximal graph, so that we can exclude the edge v_1v_9, and by symmetry also the edge v_3v_9, from the list of admissible edges for G.

Now consider the graph $G' = G+v_2v_9$. As the edge v_1v_9 is r-admissible in G', this graph is not radially-maximal. If we add an edge $e \in \{v_1v_8, v_1v_9, v_3v_8, v_3v_9\}$ to G', then the new graph contains a subgraph isomorphic to $G + v_1v_9$. This is a subcase already solved above. The remaining edges v_0v_8 and v_4v_8 are not s-admissible in G', so that also the edge v_2v_9 can be excluded from our list. As a consequence, v_9 is a vertex of degree one.

Now consider the graph $G' = G + v_0v_8$. As v_2v_9 is r-admissible edge in G', this graph is not radially-maximal. Since v_9 is a vertex of degree one, the edge v_1v_8 is the unique admissible edge in G'. However, $G' + v_1v_8$ is not a radially-maximal graph. Although out of consideration, v_2v_9 is an r-admissible edge in $G' + v_1v_8$. Hence, we can exclude the edges v_0v_8 and v_4v_8 from our list.

Finally, consider the graph $G' = G + v_1v_8$. As v_2v_9 is an r-admissible edge in G', this graph is not radially-maximal. By the previous cases, there is a unique admissible edge in G' which is not out of consideration, namely v_3v_8. But v_2v_9 is an r-admissible edge also in $G'+v_3v_8$, so that also the edges v_1v_8 and v_3v_8 can be excluded from our list. This completes the analysis of Case 1.

Case 2: Our graph does not contain the edge v_8v_9. But then v_9 is adjacent to some v_q, $0 \le q \le 7$. Denote $G = C + v_2v_8 + v_qv_9$. Since v_1v_8 is an r-admissible edge in G, this graph is not radially-maximal. There are only eight admissible edges in G, namely v_0v_8, v_1v_8, v_3v_8, v_4v_8, $v_{q-2}v_9$, $v_{q-1}v_9$, $v_{q+1}v_9$ and $v_{q+2}v_9$ (the addition is modulo 8).

Consider the graph $G' = G + v_0v_8$. Since the edge $v_{q-1}v_9$ is r-admissible in G', this graph is not radially-maximal. However, v_1v_8 is the unique admissible edge in G'. (In fact, v_1v_8 is the unique sg-admissible edge in G'.) Since $v_{q-1}v_9$ is r-admissible even in $G' + v_1v_8$, we can exclude the edge v_0v_8 from our list. And by symmetry, we can exclude also the edges v_4v_8, $v_{q-2}v_9$ and $v_{q+2}v_9$.

Thus, consider the graph $G' = G + v_1v_8$. As $v_{q-1}v_9$ is an r-admissible edge in G', this graph is not radially-maximal. Since the case $G' + v_3v_8$ reduces to the previous subcase, by symmetry, there is only one edge addable to G', namely $v_{q-1}v_9$. The graph $G' + v_{q-1}v_9$ is not selfcentric only if $q = 6$. But even in that case $G' + v_5v_9$ is not a radially-maximal graph, as v_0v_8 is an r-admissible edge in this graph. However, there are no s-admissible edges in $G' + v_5v_9$, so that also the edge v_1v_8 can be excluded from our list. And by symmetry, we can exclude also the remaining edges v_3v_8, $v_{q-1}v_9$ and $v_{q+1}v_9$, which completes the proof. $\square$

References

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