ITERATED LINE GRAPHS ARE MAXIMALLY ORDERED

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ABSTRACT. A graph G is k-ordered if for every ordered sequence of k vertices, there is a cycle in G that encounters the vertices of the sequence in the given order. We prove that if G is a connected graph distinct from a path, then there is a number t_G such that for every $t \geq t_G$ the t-iterated line graph of G, $L^t(G)$, is $(\delta(L^t(G))+1)$-ordered. Since there is no graph H which is $(\delta(H)+2)$-ordered, the result is best possible.

1. INTRODUCTION AND RESULTS

Let G be a graph. Its *line graph* $L(G)$ is defined as the graph whose vertices are the edges of G, with two vertices adjacent if and only if the corresponding edges are adjacent in G. Although the line graph operator is one of the most natural ones, only in recent years there is recorded a larger interest in studying iterated line graphs. *Iterated line graphs* are defined inductively as follows:

$$L^t(G) = \begin{cases} G & \text{if } t = 0, \\ L(L^{t-1}(G)) & \text{if } t > 0. \end{cases}$$

In iterated line graphs the greatest attention was devoted to Hamiltonicity. The most recent results in this area can be found in a paper by Xiong and Liu [16]. The diameter and radius of iterated line graphs are examined in [14], and [11] is devoted to the centres of these graphs. In [8] and [7], Hartke and Higgins study the growth

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of the minimum and the maximum degree of iterated line graphs, respectively. The connectivity of iterated line graphs is discussed in [10].

A graph G is k-ordered, if for every sequence $Z = (z_1, z_2, \ldots, z_k)$ of k distinct vertices in G, there exists a cycle that contains all the vertices of Z in the designated order. In other words, a graph is k-ordered if for every sequence $Z = (z_1, z_2, \ldots, z_k)$, there are k paths $z_1{-}z_2$, $z_2{-}z_3$, $\ldots$, $z_{k-1}{-}z_k$, $z_k{-}z_1$, that are internally-vertex-disjoint. After Chartrand introduced the notion of k-ordered graphs, several authors studied sufficient degree conditions forcing a graph to be k-ordered, see [13], [9], [4], [6], [3], [2] and [12]. Most of the papers deal with conditions based on the minimum sum of degrees of nonadjacent vertices, but for the minimum degree δ the following theorem can be found in [12].

Theorem 1. *Every graph G of order $n \geq 5k$ with $\delta(G) \geq \lceil \frac{n}{2} \rceil + \lfloor \frac{k}{2} \rfloor - 1$ is k-ordered.*

For balanced bipartite graphs the degree condition can be slacked to $\delta(G) \geq \frac{n+k-1}{4}$, provided that for the order n we have $n \geq 618$ and $3 \leq k \leq \frac{n}{206}$, see [5].

Let z_0 be a vertex of minimum degree δ in a non-complete graph G. Denote by z_1, z_2, $\ldots$, z_δ the neighbours of z_0 and choose a vertex $z_{\delta+1}$ at distance two from z_0. Then for the sequence $Z = (z_0, z_1, \ldots, z_{\delta+1})$ there is no cycle containing all the vertices of Z in the designated order. Since complete graphs K_n have only $\delta(K_n) + 1$ vertices, the necessary degree condition for a graph to be k-ordered is $\delta(G) \geq k - 1$. In this paper we show that for iterated line graphs this necessary condition is also sufficient. We prove the following theorem.

Theorem 2. *Let G be a connected graph distinct from a path. Then there is a t_G such that for every t, $t \geq t_G$, the iterated line graph $L^t(G)$ of G is $(\delta(L^t(G))+1)$-ordered.*

We remark that although the minimum degree of $L^t(G)$ grows exponentially, as a function of t, the number of vertices grow doubly exponentially (see [14] for the bounds on the number of vertices of $L^t(G)$). Hence, the minimum degree is very small compared to the number of vertices in iterated line graphs. Therefore, from the point of view of Theorem 1, it may be surprising that iterated line graphs are maximally ordered.

2. Connectivity

Let $G = (V(G), E(G))$ be a graph and let v be a vertex of $L^t(G)$, $t \geq 1$. Then v corresponds to an edge of $L^{t-1}(G)$, and this edge will be called 1-*history* of v. For $i \geq 2$ we define i-histories recursively. The i-*history* of v is a subgraph of $L^{t-i}(G)$, edges of which are induced by the vertices of $L^{t-i+1}(G)$ which are in $(i-1)$-history of v. The i-history of v is denoted by $B^i(v)$.

Observe that 1-history is always an edge and 2-history is a path of length 2. For this reason, if $v \in V(L^t(G))$ we write $B^1(v) = (v_1, v_2)$, where v_1v_2 is the edge of $L^{t-1}(G)$ corresponding to v, and $B^2(v) = (v_3, v_4, v_5)$, where v_3, v_4, v_5 is the path of length 2 in $L^{t-2}(G)$ corresponding to v.

To distinguish 2-histories from other paths in G we denote the paths without parentheses; i.e., by $P = v_1, v_2, v_3$ we denote a $v_1{-}v_3$ path of length two. This enables us to write an extension of P, by v_0 at the beginning and by v_4 at the end, as v_0, P, v_4.

A graph G is k-connected if it has at least $k{+}1$ vertices and for every pair of distinct vertices, say u and v, of G there are k internally-vertex-disjoint paths connecting u with v. Another definition, equivalent with the previous one, says that a graph G is k-connected if and only if it has at least $k+1$ vertices and for any two sets S_1 and S_2, each of k vertices, there are k vertex-disjoint paths connecting S_1 with S_2. Here we use a slightly different definition of connectivity, which is equivalent with the previous ones.

A collection of vertices $\{r_1, r_2, \ldots, r_k\}$ (not necessarily distinct) is called a multiset with k (labelled) vertices. Let $R_1 = \{r_{1,1}, r_{1,2}, \ldots, r_{1,k}\}$ and $R_2 = \{r_{2,1}, r_{2,2}, \ldots, r_{2,k}\}$ be multisets of vertices in a graph G. By k internally-vertex-disjoint paths connecting R_1 with R_2 we mean a collection $\mathcal{P}$ of internally-vertex-disjoint paths, such that exactly one path of $\mathcal{P}$ starts (terminates) in a vertex labelled by $r_{1,i}$ ($r_{2,i}$), $1 \leq i \leq k$. Hence, if r occurs q times in R_1, then exactly q paths of $\mathcal{P}$ start with r. Analogous statement is true for endvertices of paths of $\mathcal{P}$.

Definition. *A graph G is k-connected if and only if it has at least $k+1$ vertices and for every two multisets R_1 and R_2, each with k vertices, there are k internally-vertex-disjoint paths connecting R_1 with R_2.*

Observe that this definition is equivalent with the usual one. To see this, suppose that G is k-connected in the usual sense. We may assume that R_1 and R_2 are disjoint. Moreover, we may assume that there is no edge between a vertex of R_1 and a vertex of R_2. Construct from multiset R_i a set S_i, $1 \leq i \leq 2$, such that all vertices of R_i occur in S_i, and if r occurs q_r times in R_i, then insert into S_i also $q_r - 1$ neighbours of r. Since the connectivity cannot exceed the minimum degree, we can arrange this so that S_i will contain exactly k distinct vertices. As G is k-connected, there are k vertex-disjoint paths connecting S_1 with S_2, and these paths can be extended to internally-vertex-disjoint paths connecting the multisets R_1 and R_2. (Observe that both $S_1 \cap R_2$ and $S_2 \cap R_1$ are empty.) Hence, our definition is a consequence of the usual one. The converse is obvious.

The main statement in this section is Lemma 5. In its proof we use the following assertion.

Lemma 3. *Let H contain at least 7 distinct paths of length 2. Then $|E(H)| \geq 5$.*

Proof. Since 4 edges admit only $\binom{4}{2} = 6$ pairs of edges, and consequently at most 6 paths of length 2, we have $|E(H)| \geq 5$. $\square$

For two sets, $S_1, S_2 \subseteq V(G)$, by $dist_G(S_1, S_2)$ we denote the minimum distance in G between a vertex from S_1 and a vertex from S_2. For calculating the distances in $L^t(G)$ we use the following lemma, see [14].

Lemma 4. *Let G be a connected graph, $L^t(G)$ be its iterated line graph, and let u and v be distinct vertices of $L^t(G)$. Then for any i, $0 \leq i \leq t$, if the i-histories $B^i(u)$ and $B^i(v)$ are edge-disjoint, then*

$$dist_{L^t(G)}(u, v) = dist_{L^{t-i}(G)}\big(V(B^i(u)), V(B^i(v))\big) + i.$$

If the i-histories of u and v are not edge-disjoint, then $dist_{L^t(G)}(u, v) < i$.

Lemma 5. *Let G be a 10-connected graph with the minimum degree δ. Further, let $S_1, S_2 \subseteq V(L^2(G))$, $|S_1| \geq 7$, $|S_2| \geq 7$, and let $dist_{L^2(G)}(S_1, S_2) \geq 2$. Then there are $10\delta - 50$ internally-vertex-disjoint paths connecting S_1 with S_2 in $L^2(G)$.*

Proof. Observe that two distinct vertices, say u and v, in $L^2(G)$ are adjacent if and only if $B^2(u)$ and $B^2(v)$ share an edge in common. Let $P = z_0, z_1, \ldots, z_k$ be a path in G. A path $w_0, w_1, \ldots, w_{k'}$ in $L^2(G)$ is called a *P-based path* if for every i, $0 \le i \le k'$, $B^2(w_i)$ contains an edge of P. As shown in [10], if the length of P is at least 2, then there are $\delta-1$ vertex-disjoint P-based paths in $L^2(G)$. These paths are defined as follows.

There is one special path $P_1 = w_{1,0}, w_{1,1}, \ldots, w_{1,k-2}$, where $B^2(w_{1,j}) = (z_j, z_{j+1}, z_{j+2})$. Denote by $x_{2,j}, x_{3,j}, \ldots, x_{\delta-1,j}$ $\delta-2$ vertices of G that are adjacent to z_j and distinct from z_{j-1} and z_{j+1}, $1 \le j \le k-1$. Then for $i = 2, 3, \ldots, \delta-1$ we have $P_i = w_{i,0}, w_{i,1}, \ldots, w_{i,2(k-2)+1}$, where

$$B^2(w_{i,j}) = \begin{cases} (z_{\lfloor j/2 \rfloor}, z_{\lfloor j/2 \rfloor+1}, x_{i,\lfloor j/2 \rfloor+1}), & \text{if } j \text{ is even,} \\ (x_{i,\lfloor j/2 \rfloor+1}, z_{\lfloor j/2 \rfloor+1}, z_{\lfloor j/2 \rfloor+2}), & \text{if } j \text{ is odd.} \end{cases}$$

Denote the set of these $\delta-1$ paths by $\mathcal{P}_P$.

Let x, P, y and x', P', y' be paths in G. If P and P' are vertex-disjoint, then the paths of $\mathcal{P}_{x,P,y} \cup \mathcal{P}_{x',P',y'}$ are vertex-disjoint as well. Now suppose that $P = s, s_1, \ldots$ and $P' = s, s'_1, \ldots$ are internally-vertex-disjoint sharing a common starting vertex s (assume that their endvertices are distinct). Moreover, suppose that $x \neq x'$, where neither x nor x' are in $V(P) \cup V(P')$. Then the paths of $\mathcal{P}_{x,P,y} \cup \mathcal{P}_{x',P',y'}$ are vertex-disjoint up to some exceptions. But after deleting from $\mathcal{P}_{x,P,y} \cup \mathcal{P}_{x',P',y'}$ two paths of $\mathcal{P}_{x',P',y'}$ starting with w'_1, $B^2(w'_1) = (x', s, x)$, and w'_2, $B^2(w'_2) = (x', s, s_1)$, respectively, we are left with a set of vertex-disjoint paths.

Denote by H_i the subgraph of G formed by edges of 2-histories of vertices of S_i, $1 \le i \le 2$. By Lemma 3, H_i contains at least 5 edges, say $e_{i,1}, e_{i,2}, \ldots, e_{i,5}$. Denote $e_{1,j} = x_{2j-1}x_{2j}$ and $e_{2,j} = y_{2j-1}y_{2j}$, $1 \le j \le 5$. By Lemma 4, H_1 and H_2 are edge-disjoint, as $dist_{L^2(G)}(S_1, S_2) \ge 2$. Since G is 10-connected, there are 10 internally-vertex-disjoint paths in G, say $P_1, P_2, \ldots, P_{10}$, connecting the multiset $\{x_1, x_2, \ldots, x_{10}\}$ with $\{y_1, y_2, \ldots, y_{10}\}$. Obviously, none of $P_1, P_2, \ldots, P_{10}$ uses an edge of H_i, $1 \le i \le 2$. Extend these paths, each by one vertex at the beginning and by one vertex at the end, so that the path which started with a vertex labelled by x_i will start with x_{i+1} if i is odd and with x_{i-1} if i is even, and analogously, if the path terminated with a vertex labelled by y_j it will finish with y_{j+1} if j is odd and with y_{j-1} if j is even. In such a way, every path starts with an edge of H_1 and it terminates with an edge of H_2. Denote by $Q_1, Q_2, \ldots, Q_{10}$ the set of extended paths.

Since $P_1, P_2, \ldots, P_{10}$ are internally-vertex-disjoint, the paths of $\mathcal{P}_{Q_1} \cup \mathcal{P}_{Q_2} \cup \cdots \cup \mathcal{P}_{Q_{10}}$ are vertex-disjoint up to some exceptions. By the previous analysis, with every pair of adjacent edges $e_{i,j}$ and $e_{i,j'}$ we must delete two paths. Observe that 5 edges admit at most $\binom{5}{2} = 10$ adjacent pairs of edges. Thus, deleting $2 \cdot 10$ paths will solve the situation at H_1, and deleting another $2 \cdot 10$ paths will solve the situation at H_2. Hence, in $\mathcal{P}_{Q_1} \cup \mathcal{P}_{Q_2} \cup \cdots \cup \mathcal{P}_{Q_{10}}$ there are $10(\delta-1) - 4 \cdot 10 = 10\delta - 50$ vertex-disjoint paths. For each of these paths, its starting vertex is adjacent with a vertex of S_1, and its terminal vertex is adjacent with a vertex of S_2. Hence, there are $10\delta - 50$ internally-vertex-disjoint paths connecting S_1 with S_2 in $L^2(G)$. $\square$

The next statement can be found in [10].

Theorem 6. *Let G be a connected graph with the minimum degree $\delta \ge 3$. Then $L^2(G)$ is $(\delta-1)$-connected.*

We conclude this section with a lemma, which is in a sense complementary to Lemma 5.

Lemma 7. *Let G be a graph with the minimum degree $\delta \geq 2$, and let u and v be vertices in $L^2(G)$. Then there are at least $6\delta - 12$ distinct vertices in $V(L^2(G)) - \{u, v\}$, which are adjacent to either u or v.*

Proof. Let w be a vertex of $L^2(G)$ with $B^2(w) = (w_0, w_1, w_2)$. Since $\delta(G) = \delta$, in $L^2(G)$ there are $\delta - 1$ neighbours of w with 2-history (x, w_0, w_1) for some x, and there are $\delta - 2$ neighbours of w with 2-history (w_0, w_1, x) for some x. Hence, the total number of neighbours of w in $L^2(G)$ is at least $2(2\delta - 3) = 4\delta - 6$.

Let us consider 2-histories of u and v. If $B^2(u)$ and $B^2(v)$ do not share an edge, then it is a matter of routine to check that there are at most 4 vertices which are adjacent to both u and v. Hence, the number of vertices adjacent to either u or v is at least $(4\delta - 6) + (4\delta - 6) - 4 = 8\delta - 16 \geq 6\delta - 12$.

Now suppose that $B^2(u)$ and $B^2(v)$ share an edge. Then $B^2(u) \cup B^2(v)$ forms either a path, or a triangle, or a claw $K_{1,3}$, and the neighbourhood of $\{u, v\}$ contains at least $6\delta - 10$, $6\delta - 11$, $6\delta - 12$ vertices, respectively. $\square$

Notice that $\delta(L^2(G)) \geq 4\delta(G) - 6$. If $\delta(L^2(G)) = 4\delta(G) - 6$, then by Lemma 7, for any pair of vertices $u, v \in V(L^2(G))$, there are at least $\frac{3}{2}\delta(L^2(G)) - 3$ vertices in $V(L^2(G)) - \{u, v\}$ which are adjacent to either u or v.

3. Separations

Observe that the t-iterated line graph of a path on n vertices is a path on $n - t$ vertices for $t < n$ and an empty graph if $t \geq n$. The iterated line graph of a cycle is isomorphic to the original cycle, and each iterated line graph of a claw $K_{1,3}$ is isomorphic to a triangle. Hence, it suffices to study connected graphs distinct from paths, cycles and the claw $K_{1,3}$. Such graphs are called *prolific*, since every two members of the sequence $\{L^t(G)\}_{t=0}^{\infty}$ are non-isomorphic.

We are interested in t-iterated line graphs when t is 'big enough'. For these graphs Hartke and Higgins in [8] proved the following theorem.

Theorem 8. *Let G be a prolific graph. Then there is an i_G such that for every i, $i \geq i_G$, we have*

$$\delta(L^{i+1}(G)) = 2 \cdot \delta(L^i(G)) - 2.$$

A *separation* of a graph G is a pair (A, B) of subsets of $V(G)$, such that $A \cup B = V(G)$ and there is no edge in G joining a vertex of $A - B$ with a vertex of $B - A$. The order of separation (A, B) is $|A \cap B|$. In [15] there is the following theorem.

Theorem 9. *Let G be a graph and let $Z \subseteq V(G)$. Let $m \geq \left\lfloor \frac{3}{2}|Z| \right\rfloor$ and let $G_1, G_2, \ldots, G_m$ be connected subgraphs of G, mutually vertex-disjoint, such that for $1 \leq i < j \leq m$ there is an edge of G between G_i and G_j. Suppose that there is no separation (A, B) of G of order $< |Z|$ with $Z \subseteq A$ and $A \cap V(G_i) = \emptyset$ for some i $(1 \leq i \leq m)$. Then for every partition $(Z_1, Z_2, \ldots, Z_n)$ of Z into non-empty subsets, there are n connected subgraphs $T_1, T_2, \ldots, T_n$ of G, mutually disjoint and with $V(T_j) \cap Z = Z_j$ $(1 \leq j \leq n)$.*

The next statement can be found in [1].

Lemma 10. *Let G be a graph with the minimum degree $\delta \geq 5$, and $m = \delta \cdot \lfloor \sqrt{\delta - 1} \rfloor$. Then there is a subgraph K of $L^3(G)$, such that K_m is a minor of K.*

In the proof of Theorem 2 we examine separations which separate a set Z of $\delta(L^t(G)) + 1$ vertices from a subgraph K mentioned in Lemma 10.

Proof of Theorem 2. Since the statement of Theorem 2 is obvious for cycles and the claw $K_{1,3}$, assume that G is a prolific graph. Denote $\delta_r = \delta(L^{t-r}(G))$. Suppose that t is so big that $\delta_4 \geq 11$, $\delta_3 \geq 577$ and $t \geq i_G + 3$, where i_G is the constant from Theorem 8.

By Lemma 10, there is a subgraph K of $L^t(G)$ such that K_m is a minor of K, $m = \delta_3 \cdot \lfloor \sqrt{\delta_3 - 1} \rfloor$. As $t \geq i_G + 3$, we have $\delta_0 = 8\delta_3 - 14$ by Theorem 8. Since $\delta_3 \geq 577$, we have

$$m = \delta_3 \cdot \lfloor \sqrt{\delta_3 - 1} \rfloor \geq 24\delta_3 > 3(\delta_0 + 1) = 3|Z|.$$

Denote by G_i, $1 \leq i \leq m$, the subgraphs of K which form the vertices of K_m when contracted into single vertices. We may assume that every G_i is so large that it is connected to every G_j, $j \neq i$, by an edge. In [1] there is described a construction of the graph K. The number of vertices in each of the subgraphs G_i is much greater than 7 (in fact $|V(G_i)| \geq 24 + 24 \cdot (\delta_3 - 1)$). The construction is rather technical, so that we do not repeat it here.

Let $H = L^t(G)$. Denote by $Z = \{z_0, z_1, \ldots, z_{\delta_0}\}$ the set of δ_0+1 vertices which have to be traversed in a given order. Assume that these vertices are labelled so that the required cycle must pass them in order $z_0, z_1, \ldots, z_{\delta_0}$. Replace every vertex $z_i \in Z$ by pair of adjacent vertices z_i^- and z_i^+ which are connected by an edge to every neighbour of z_i in H, and denote the resulting graph by H^*. We show that H^* contains vertex-disjoint paths connecting z_i^+ with z_{i+1}^-, $0 \leq i \leq \delta_0$ (the indices are taken modulo δ_0+1). To do this, it is sufficient to show that there is no separation (A^*, B^*) of order $< 2(\delta_0+1)$ in H^*, such that $Z^* = \{z_0^-, z_0^+, \ldots, z_{\delta_0}^-, z_{\delta_0}^+\} \subseteq A^*$ and $V(G_i) \subseteq B^* - A^*$ for some i, $1 \leq i \leq \frac{3}{2}|Z^*| = 3|Z|$, by Theorem 9. In the rest of the proof we call a separation of order $< |Z^*|$ a bad one.

Suppose that there is a bad separation. If one of z_i^-, z_i^+ belongs to $A^* - B^*$, then we may assume that both of them do, since these vertices have identical neighbourhood in $H^* - \{z_i^-, z_i^+\}$. Hence, (A^*, B^*) induces a separation (A, B) of H.

First suppose that for the induced separation (A, B) we have $|A - B| \geq 7$. Since $\delta_4 \geq 11$, the graph L^{t-2} is 10-connected, by Theorem 6. For every G_i with $V(G_i) \subseteq B - A$ we have $dist_{L^t(G)}(V(G_i), A{-}B) \geq 2$, and also $|V(G_i)| \geq 7$ and $|A - B| \geq 7$. Hence, applying Lemma 5 to $L^{t-2}(G)$ we get $|A \cap B| \geq 10\delta_2 - 50 \geq 8\delta_2 - 10$. (Observe that $\delta_2 \geq 20$ follows from $\delta_3 \geq 577$.) Since $t > i_G + 2$, we have $\delta_0 = 4\delta_2 - 6$, so that $8\delta_2 - 10 = 2(\delta_0+1)$. Hence $|A^* \cap B^*| \geq |A \cap B| \geq 2(\delta_0+1) = |Z^*|$, so that the separation cannot be bad.

Now suppose that $2 \leq |A - B| \leq 6$. The neighbourhood of two vertices from $A - B$ contains at least $6\delta_2 - 12$ vertices by Lemma 7. As $|A - B| \leq 6$, we have $|A \cap B| \geq 6\delta_2 - 12 - 4 = 6\delta_2 - 16$. Since $t > i_G + 2$, we have $\delta_0 = 4\delta_2 - 6$, which implies that $A \cap B$ contains at least $\delta_0 - 5 = 4\delta_2 - 11$ vertices from Z. As the vertices of Z are 'doubled' in H^*, we have $|A^* \cap B^*| \geq 6\delta_2 - 16 + 4\delta_2 - 11 = 10\delta_2 - 27 > 8\delta_2 - 10 = 2(\delta_0+1)$. (Observe that $\delta_2 \geq 9$ follows from $\delta_3 \geq 577$.) Thus, also in this case the separation cannot be bad.

If $(A - B) \cap Z = \emptyset$ then the separation cannot be bad as $Z \subseteq A \cap B$, and consequently $|A^* \cap B^*| \geq 2(\delta_0+1)$. Thus, we may assume that $A - B = \{z_0\}$. If the neighbourhood of z_0 contains at most $\delta_0 - 2$ vertices of Z, then it contains at least two vertices outside Z, so that $|A^* \cap B^*| \geq 2\delta_0 + 2$. Thus, the neighbourhood of z_0 contains at least $\delta_0 - 1$ vertices from Z. Denote by N the set of neighbours of z_0 in H. There are two cases to distinguish:

Case 1: $|A^* \cap B^*| = 2\delta_0$. In this case the separation is bad, so that we cannot apply Theorem 9 on Z^* in H^*. However, now $N = Z - \{z_0\}$, so that there are edges $z_{\delta_0}z_0$ and z_0z_1 in H. Thus, it is sufficient to find internally-vertex-disjoint paths $z_1{-}z_2$, $z_2{-}z_3$, ..., $z_{\delta_0-1}{-}z_{\delta_0}$ in H. To do this, we consider a new graph H'.

Let H' be obtained from H^* by contracting the edges $z_{\delta_0}^- z_{\delta_0}^+$ and $z_1^- z_1^+$, and by deleting the vertices z_0^- and z_0^+. Further, let $Z' = \{z_1, z_2^-, z_2^+, \dots, z_{\delta_0-1}^-, z_{\delta_0-1}^+, z_{\delta_0}\}$. Since we have deleted just two vertices from the graph, it contains a subgraph K' such that K_{m-2} is a minor of K'. However, $m-2 \geq 3(\delta_0+1)-2 > 3(\delta_0-1) = \frac{3}{2}|Z'|$. Hence H' contains, as a minor, a 'sufficiently big' complete graph.

By Theorem 9, it is sufficient to show that there is no separation (A', B') of order $< |Z'| = 2(\delta_0-1)$ in H', such that $Z' \subseteq A'$ and $V(G_i) \subseteq B' - A'$ for some i, $1 \leq i \leq \frac{3}{2}|Z'|$. Analogously as above, denote by (A, B) a separation in H induced by the separation (A', B') in H'. We remark that for the induced separation (A, B) we always set $z_0 \in A \cap B$. In the next we consider separations (A', B').

First suppose that for the induced separation (A, B) we have $|A - B| \geq 7$. Then $|A \cap B| \geq 10\delta_2 - 50$, and analogously as above we get $|A' \cap B'| \geq |A \cap B| - 2 \geq 2(\delta_0-1) = |Z'|$.

Now suppose that $|A - B| \leq 6$. Since (A', B') cannot be bad if $(A - B) \cap Z = \emptyset$, suppose that there is $z' \in A' - B'$, where $z' \in Z'$. Denote by z the vertex of H corresponding to z'. By Lemma 7, there are at least $6\delta_2 - 12$ vertices in $V(H) - \{z_0, z\}$, which are adjacent to either z_0 or z. As a consequence, there are at least $6\delta_2 - 12 - (\delta_0 - 1) = 2\delta_2 - 5$ vertices from $V(H) - Z$ which are adjacent to z. But then $|A' \cap B'| \geq 2(\delta_0-1) - 6 + 2\delta_2 - 5 \geq 2(\delta_0-1) = |Z'|$.

Hence, there is no bad separation in H', so that by Theorem 9 there are vertex-disjoint paths $z_1{-}z_2^-$, $z_2^+{-}z_3^-$, ..., $z_{\delta_0-1}^+{-}z_{\delta_0}$ in H'.

Case 2: $|A^* \cap B^*| = 2\delta_0 + 1$. Also in this case the separation is bad, so that we cannot apply Theorem 9 on Z^* in H^*. But since N contains at least $\delta_0 - 1$ vertices from A, either $z_{\delta_0}z_0$ or z_0z_1 is in H. By symmetry, we may assume that $z_0z_1 \in E(H)$.

Consider a graph H' obtained from H^* by contracting the edges $z_0^- z_0^+$ and $z_1^- z_1^+$, with $Z' = \{z_0, z_1, z_2^-, z_2^+, \dots, z_{\delta_0}^-, z_{\delta_0}^+\}$. By Theorem 9, it is sufficient to show that there is no separation (A', B') of order $< |Z'| = 2\delta_0$ in H', such that $Z' \subseteq A'$ and $V(G_i) \subseteq B' - A'$ for some i, $1 \leq i \leq \frac{3}{2}|Z'|$.

However, for the separation (A', B') with $A' - B' = \{z_0\}$ we have $|A' \cap B'| = 2\delta_0$, since $|A^* \cap B^*| = 2\delta_0 + 1$. And analogously as in the previous case one can show that the other separations are not bad. Hence, there is no bad separation in H'.

By Theorem 9, there is a cycle in H traversing the vertices of Z in the prescribed order. As the only assumption on Z was $|Z| = \delta_0 + 1$, the graph $H = L^t(G)$ is (δ_0+1)-ordered. $\square$

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