Note

Regular hamiltonian embeddings of $K_{n,n}$ and regular triangular embeddings of $K_{n,n,n}$

Martin Knor and Jozef Širáň

Department of Mathematics, SvF
Slovak University of Technology
Bratislava, Slovakia

Abstract

We give a group-theoretic proof of the following fact, proved initially by methods of topological design theory: Up to isomorphism, the number of regular hamiltonian embeddings of $K_{n,n}$ is 2 or 1, depending on whether n is a multiple of 8 or not. We also show that for each n there is, up to isomorphism, a unique regular triangular embedding of $K_{n,n,n}$.

1 Introduction

A 2-cell embedding of a graph on a surface (orientable or not) is said to be *regular* if the automorphism group of the embedding acts transitively, and hence regularly, on the flags (which, in this note, can be identified with mutually incident vertex-edge-face triples) of the embedding. If an embedding of a graph in an orientable surface admits a group of automorphisms acting transitively on the darts (edges with direction) of the graph, we say that the embedding is *orientably regular*.

The study of orientably regular and regular embeddings has a rich history and is closely related to group theory, hyperbolic geometry, theory of Riemann surfaces, and Galois theory; see [11] for a survey. The quest for classification of regular embeddings takes three natural directions: by automorphism groups, by supporting surfaces, and by embedded graphs. Except for [9, 14], little is known in the first direction. For recent progress in the second direction we refer to [3, 1]. As regards the third direction, the only important class of graphs for which orientably regular and regular embeddings have been classified by the time of submission of this article are complete graphs [8, 15]. However, significant results have lately been obtained for the next two natural candidate classes, the complete bipartite graphs $K_{n,n}$ (see e.g. [10]) and the n-cubes (see e.g. [4]), in both cases extending earlier results of [12]. We remark that it is easy to prove uniqueness of orientably regular embeddings of $K_{p,p}$ where p is a prime [13]; face boundaries in these embeddings are hamiltonian cycles.

In this note we extend the result of [13] by classifying, for all n, the regular embeddings of $K_{n,n}$ in which faces are bounded by closed walks of length at least $2n$. It turns out that in such maps, faces boundaries must be hamiltonian cycles (and we speak about hamiltonian embeddings). We also prove that, up to isomorphism, for each n the graph $K_{n,n,n}$ admits a unique regular embedding with triangular faces. We note that uniqueness of orientably regular embeddings of the graphs $K_{p,p,p}$ for prime p follows also from [5]. All these embeddings are forcibly orientable.

Special cases of the above results have, rather surprisingly, been derived first by methods of topological design theory, with restriction to orientable embeddings [6, 7]. The purpose of this note is to supply completely different proofs based on elementary group theory.

2 Regular hamiltonian embeddings of $K_{n,n}$

Let e be an edge of a regular embedding M of $K_{n,n}$ in some surface, with face boundaries of length $2l \geq 2n$. The automorphism group G_n of M then contains an element r of order n which rotates the map about a vertex incident with e. Similarly, G_n contains commuting involutions a, b such that a rotates the map about the center of the edge e while b reflects M in e. It is well known that $G_n = \langle a, b, r \rangle$; by regularity, $|G_n| = 4n^2$. The conjugate $s = ara$ is a rotation of M about the other vertex incident with e. The automorphisms r, a, s rotate the map in the same local direction (referring to some open neighbourhood of e) while b reverses the local direction. We also have $brb = r^{-1}$ and $bsb = s^{-1}$.

The subgroup $H_n = \langle r, s, b\rangle$ with $|H_n| = 2n^2$ preserves the natural two-colouring of vertices of $K_{n,n}$; clearly $G_n = H_n \rtimes \langle a\rangle$. The element rs, of order $l \geq n$, is a colour preserving rotation of a face of M incident with e. Since r and s fix vertices in different parts, $\langle r\rangle \cap \langle s, b\rangle = 1$. Also, it can be checked that the cyclic group $\langle rs\rangle$ intersects the dihedral groups $\langle r, b\rangle$ and $\langle s, b\rangle$ just trivially if $n \geq 3$. Therefore for $n \geq 3$ the group G_n has the form $G_n = H_n \rtimes \langle a\rangle$ where $H_n = \langle r\rangle\langle s, b\rangle = \langle r, s, b |\ r^n = s^n = (rs)^n = b^2 = (br)^2 = (bs)^2 = \ldots = 1\rangle$. The supporting surface for M is orientable if and only if $b \notin \langle r, s\rangle$, and then $H_n = \langle r\rangle\langle s\rangle \rtimes \langle b\rangle$.

Omitting the trivial spherical embedding of $K_{1,1}$, the graph $K_{2,2}$ admits two regular embeddings: a hamiltonian one in a sphere and a projective-planar one with face length 8. If $n \geq 3$, then from $\langle rs\rangle \cap \langle r, b\rangle = 1$ we obtain $2n^2 = |H_n| \geq |\langle rs\rangle| \cdot |\langle r, b\rangle| = l \cdot 2n$, which implies that $l \leq n$; since we have assumed that $l \geq n$, we conclude that $l = n$. In other words, if $n \geq 3$, a regular embedding of $K_{n,n}$ of face length at least $2n$ must be hamiltonian. For such embeddings we have the following classification result.

Theorem 1 *If $n \geq 3$, the graph $K_{n,n}$ admits exactly two regular hamiltonian embeddings if n is a multiple of 8, and a unique regular hamiltonian embedding in all other cases; the embeddings are necessarily orientable.*

Proof. Since $H_n = \langle r\rangle\langle s, b\rangle$ with $\langle r\rangle \cap \langle s, b\rangle = 1$, for each $i \in [n] = \{0, 1, \ldots, n-1\}$ we have $(rs)^i = r^{k_i}s^{l_i}b^{\varepsilon_i}$ for unique $k_i, l_i \in [n]$ and $\varepsilon_i \in \{0, 1\}$; conjugation by ab yields $(s^{-1}r^{-1})^i = s^{-k_i}r^{-l_i}b^{\varepsilon_i}$. Let $\delta_i = (-1)^{\varepsilon_i}$. Inverting both sides of the last equation and rearranging terms gives $(rs)^i = r^{\delta_i l_i}s^{\delta_i k_i}b^{\varepsilon_i}$. We see that $l_i = \delta_i k_i$, and so for each $i \in [n]$ we have

$$(rs)^i = r^{k_i}s^{\delta_i k_i}b^{\varepsilon_i}\ . \tag{1}$$

Conjugation of (1) by ra leaves the left-hand side of (1) invariant and transforms the right-hand side into $(rs)^i = s^{k_i-1}r^{\delta_i k_i}s^{\delta_i}b^{\varepsilon_i}$. Comparison with (1) then gives, for each $i \in [n]$,

$$r^{k_i}s^{\delta_i(k_i-1)} = s^{k_i-1}r^{\delta_i k_i}\ . \tag{2}$$

Suppose that $k_j = k_i$ for some $j \neq i$. Then we have $(rs)^j = r^{k_i}s^{\delta_j k_i}b^{\varepsilon_j}$. which, after left multiplication by the inverse of (1) and rearrangement yields

$(rs)^{j-i} = s^{(\delta_i\delta_j-1)k_i}b^{\varepsilon_i+\varepsilon_j}$. Since $j \neq i$, it follows that $\varepsilon_j \neq \varepsilon_i$ and hence $\delta_j \neq \delta_i$, giving $(rs)^{j-i} = s^{-2k_i}b$. This, however, contradicts the fact that $\langle rs\rangle \cap \langle s, b\rangle = 1$. We conclude that the n exponents $k_i \in [n]$ are distinct. Hence there exists a unique $\ell \in [n]$ such that $k_\ell = 2$.

Suppose first that $\varepsilon_\ell = 1$. Note that now $b \in \langle r, s\rangle$ and the embedding is nonorientable. Then, (2) with $k_\ell = 2$ and $\delta_\ell = -1$ gives $r^2s^{-1} = sr^{-2}$ and conjugation by a results in $s^2r^{-1} = rs^{-2}$. If $n = 3$ this reduces to $(rs)^2 = 1$, contradiction. For $n \geq 4$ let us take (2) again but this time for the (unique) element $i \in [n]$ such that $k_i = 3$, that is, $r^3s^{2\delta_i} = s^2r^{3\delta_i}$. Multiplying the two identities and rearranging terms we arrive at $s^2r^2 = r^{1+3\delta_i}s^{-2\delta_i}$. Taking the ab conjugate and then inverting both sides leaves the left-hand side of the last identity unaltered and yields $s^2r^2 = r^{-2\delta_i}s^{1+3\delta_i}$. Comparing right-hand sides of the last two equalities leads to $r^{1+5\delta_i} = s^{1+5\delta_i}$. Since we have assumed that $n \geq 4$ and $\langle r\rangle \cap \langle s\rangle = 1$, it follows that $n = 4$ if $\delta_i = -1$ and $n = 6$ if $\delta_i = 1$. The catalog of small regular maps of [3] shows, however, that there do not exist nonorientable regular hamiltonian embeddings of $K_{4,4}$ and $K_{6,6}$. We note that the only arithmetically feasible candidate for $n = 6$, the dual of the map N20.3 of [3], has a non-bipartite underlying graph.

The preceding analysis shows that $\varepsilon_\ell = 0$. By (2), $r^2s = sr^2$. If $b \in \langle r, s\rangle$, then the last identity shows that r^2 is in the center of H_n; in particular, $r^2b = br^2$. But conjugation by b also satisfies $r^2b = br^{-2}$, which then implies that $r^4 = 1$ and leads to a contradiction as above. We are therefore left with $b \notin \langle r, s\rangle$, that is, $H_n = H_n^* \rtimes \langle b\rangle$ where $H_n^* = \langle r\rangle\langle s\rangle$. This also shows that there are no regular hamiltonian embeddings of $K_{n,n}$ in nonorientable surfaces for any n. Conjugating the identity $r^2s = sr^2$ by $a \in G_n$ we obtain $s^2r = rs^2$, which implies that $\langle r^2, s^2\rangle$ is a central subgroup of H_n^*.

Let $k_2 = j$, so that $(rs)^2 = r^js^j$. It follows that $sr = r^{j-1}s^{j-1}$ and, by symmetry, $rs = s^{j-1}r^{j-1}$. If j was odd then s^{j-1} and r^{j-1} would commute, which would imply $rs = sr$ and $j = 2$, contradiction. Therefore j must be even. Still using centrality of $\langle r^2, s^2\rangle$ in H_n^* and conjugation by $a \in G_n$ we obtain $(rs)^2 = r^js^j = s^jr^j = (sr)^2$ and $(rs)^2 = (sr)^2 = sr \cdot sr = r^{j-1}s^{j-1} \cdot r^{j-1}s^{j-1} = r^{j-1}s^{j-1} \cdot r^{j-2}s^{j-2}rs = r^{2j-3}s^{2j-3}rs$. After cancellation we are left with $rs = r^{2j-3}s^{2j-3}$, which is possible only if $2j-3 \equiv 1 \pmod{n}$ or equivalently $2j-4 \equiv 0 \pmod{n}$. Since $0 \leq j < n$, the last equation has only two solutions, namely $j = 2$ and $j = 2 + n/2$. The first solution gives $rs = sr$ and hence $H_n^* = \langle r\rangle \times \langle s\rangle \cong Z_n \times Z_n$, with the obvious presentation $H_n^* = \langle r, s \,|\, r^n = s^n = (rs)^n = [r, s] = 1\rangle$ where $[r, s] = r^{-1}s^{-1}rs$.

We will now analyze the second solution and identify the corresponding group. Here j is even and therefore n is a multiple of 4; say, $n = 4d$, and $j = 2 + 2d$. Using centrality of even powers of r and s, with the involutory element $t = r^{2d}s^{2d}$ we rewrite $(rs)^2 = r^j s^j$ in the form $(rs)^2 = r^2s^2t$, which simplifies further to $t = [r, s]$ (and also to $t = [s, r]$ since $t^2 = 1$). Raising the last identity for $(rs)^2$ to the power of d we obtain $(rs)^{2d} = r^{2d}s^{2d}t^d = t^{1+d}$. Since the order of rs is $4d$ it follows that t^{1+d} is a non-trivial involution, which means that d must be even and hence n must be divisible by 8. By the facts appearing at the beginning of the proof, $t = r^{2d}s^{2d}$ is also a power of rs. But there is only one such non-trivial involutory power and therefore $(rs)^{2d} = t = [s, r]$, which implies that $sr = r^{-1}(rs)r = (rs)^{2d+1}$. Summing up, the second solution $j = 2+n/2$ implies that $8|n$ and that $H_n^* = \langle rs\rangle \rtimes \langle r\rangle \cong Z_n \rtimes Z_n$ where the semidirect product is determined by the above conjugation identity. In terms of a presentation, $H_n^* = \langle r, s|\ r^n = s^n = (rs)^n = (rs)^{n/2}[r, s] = 1\rangle$.

In both cases the group H_n^* of order n^2 has two automorphisms of order 2 interchanging r with s and inverting both r and s, respectively. This completely determines G_n in the form $G_n = H_n^* \rtimes \langle a, b\rangle$ where a, b are commuting involutions representing the two automorphisms of H_n. Moreover, in each case the split extension of H_n^* by $\langle a\rangle$ is a subgroup of G_n of index two consisting precisely of the orientation preserving automorphisms. By general theory of maps (see e.g. [2]), up to isomorphism these are the only regular hamiltonian embeddings of $K_{n,n}$, since any such embedding with rotations r', s' and involutions a', b' would lead to the same presentation of the automorphism group as obtained in the course of the proof, with r', s', a', b' in place of r, s, a, b, giving isomorphic embeddings. □

3 Regular triangular embeddings of $K_{n,n,n}$

On the basis of Theorem 1 it is now relatively easy to classify regular triangular embeddings of complete tripartite graphs with equal parts.

Theorem 2 *The graph $K_{n,n,n}$ admits a unique regular triangular embedding for any n; the embedding is orientable.*

Proof. It suffices to assume that $n \geq 3$. Let F_n be the automorphism group of a regular triangular embedding M' of the graph $K_{n,n,n}$ in some surface. We may assume that vertices of $K_{n,n,n}$ have been properly 3-coloured, say, black, white and green. It is clear that deletion of the n green vertices from M' yields a regular hamiltonian embedding of $K_{n,n}$ whose automorphism group G_n is a subgroup of F_n consisting of all automorphisms in F_n that preserve green vertices setwise. By Theorem 1 and its proof we may assume that the supporting surface of the embedding is orientable and that $G_n \cong H_n^* \rtimes \langle a, b\rangle$ where $H_n^* = \langle r\rangle\langle s\rangle$ and the conjugation actions by a, b are as described before.

Let τ be a fixed triangular face of M'. Then r, s, and $(rs)^{-1}$ can be assumed to represent the colour-preserving rotations of M', by two triangles each, about the black, white, and green vertex of τ, respectively, so that the rotations are clockwise with respect to τ. Similarly, a and b can be assumed to represent the half-turn of M' about the centre of the edge of τ lying opposite the green vertex and the reflection of M' in this edge, respectively.

By regularity of M' the three rotations are conjugate via an element $c \in F_n$ of order 3 which represents a clockwise (with respect to τ) rotation of M' about the center of the face τ; that is, $csc^{-1} = r$, $crc^{-1} = (rs)^{-1}$ and $c(rs)^{-1}c^{-1} = s$. Further, one can check by following the action of the automorphisms a, b, c, r on the 'corner' x of τ containing the black vertex that $(ca)^2(x) = cbc^{-1}b(x) = r^{-1}(x)$. Invoking regularity of M' again, it follows that $(ca)^2 = cbc^{-1}b = r^{-1}$. The two identities will be tacitly used at the end of the proof in their equivalent forms $cac^{-1} = r^{-1}ac$ and $cbc^{-1} = br$; note that the last one can be rewritten in the form $bcb = rc$.

With the extra information about c we can now specify H_n^*. Indeed, in the proof of Theorem 1 we saw that $(rs)^2 = r^j s^j$ for some j. Conjugating this identity by c and rearranging terms yields $(rs)^j = r^j s^2$, which (by the previous proof again) is possible if and only if $j = 2$. It follows that in this case for each n we have $H_n^* = \langle r\rangle \times \langle s\rangle$ and, of course, $G_n = H_n^* \rtimes \langle a, b\rangle$ as before.

The entire group F_n is now uniquely determined. Since all elements of G_n preserve the set of green vertices while c and c^2 do not, we have $G_n \cap \langle c\rangle = 1$. This together with the facts that $|F_n| = 12n^2$ and $|G_n| = 4n^2$ implies that $F_n = G_n\langle c\rangle$. The way c acts on r, s, a, b is given by the identities derived above. Finally, note that F_n also admits a description in the form $F_n = F_n^{\#} \rtimes \langle b\rangle$ where $F_n^{\#} = (H_n^* \rtimes \langle a\rangle)\langle c\rangle$ is the group of orientation preserving automorphisms of M'. Uniqueness of the regular triangular embedding of $K_{n,n,n}$ now follows by the same argument as given at the end of the proof of Theorem 1. $\square$

4 Remarks

In the theory of regular embeddings on orientable surfaces it is customary to describe orientation preserving automorphism groups in terms of two generators R and L representing the automorphisms that rotate the embedding (say, clockwise) about a vertex (by one face) and about the center of an edge incident with the vertex. In order to obtain this type of presentation for our group $F_n^{\#}$ we may take $L = a$ and $R = ac^{-1}$. With the help of the relations established in the course of the previous proofs one can check that $r = R^2$ and $s = LR^2L$. Further, $[r,s] = 1$ turns out to be equivalent with $[L,R]^3 = 1$. This identity also implies that $crc^{-1} = (rs)^{-1}$ with $c = R^{-1}a = R^{-1}L$ and with r,s as above (we note that in this setting we have $csc^{-1} = r$ for free). Consequently, the full presentation of the orientation preserving automorphism group of the unique regular triangular embedding of $K_{n,n,n}$ can be given in the form $F_n^{\#} = \langle R, L |\ R^{2n} = L^2 = (LR)^3 = [L,R]^3 = 1\rangle$. A result of the same type for the orientation preserving automorphism groups $G_n^{\#} = H_n^* \rtimes \langle a\rangle$ of the regular hamiltonian embeddings of $K_{n,n}$ described in Theorem 1 follows by directly taking the rotations r and a for generators. A much shorter computation then gives $G_n^{\#} = \langle r, a|\ r^n = a^2 = (ra)^{2n} = (ar)^2(ra)^{\delta n-2} = 1\rangle$ where $\delta \in \{0,1\}$ if $8|n$ and $\delta = 0$ otherwise. For completeness we recall that the full automorphism groups F_n and G_n are semidirect products of $F_n^{\#}$ and $G_n^{\#}$ by $\langle b\rangle$, where conjugation by b inverts all the above generators.

Acknowledgment The research of both authors was supported by the VEGA Grant No. 1/2004/05 and by the APVV Grant No. 20-000704. Research of the second author was also sponsored by the U.S.-Slovak Science and Technology Joint Fund under Project Number 020/2001.

References

[1] A. Breda, R. Nedela, J. Širáň, Classification of regular maps of negative prime Euler characteristic, *Trans. Amer. Math. Soc.* **357** (2005) No 10, 4175-4190.

[2] R. P. Bryant, D. Singerman, Foundations of the theory of maps on surfaces with boundary. *Quart. J. Math. Oxford Ser.* (2) **36** (1985), No. 141, 17-41.

[3] M. Conder, P. Dobcsányi, Determination of all regular maps of small genus. *J. Combinat. Theory Ser. B* **81** (2001) No. 2, 224-242.

[4] S. F. Du, J. H. Kwak, R. Nedela, Classification of regular embeddings of hypercubes of odd dimension, *Discrete Math.* **307** (2007) No. 1, 119-124.

[5] S. F. Du, J. H. Kwak, R. Nedela, Regular embeddings of complete multipartite graphs, *European J. Combin.* **26** (2005) No. 3-4, 505-519.

[6] M. J. Grannell, T.S. Griggs and M. Knor, Regular Hamiltonian embeddings of the complete bipartite graph $K_{n,n}$ in an orientable surface, *Congressus Numerantium* **163** (2003), 197-205.

[7] M. J. Grannell, T.S. Griggs, M. Knor and J. Širáň, Triangulations of orientable surfaces by complete tripartite graphs, *Discrete Math.* **306** (2006) No. 6, 600-606.

[8] L. D. James, G. A. Jones, Regular orientable imbeddings of complete graphs, *J. Combinat. Theory Ser. B* **39** (1985), 353-367.

[9] G. A. Jones, Ree groups and Riemann surfaces, *J. Algebra* **165** (1994) No. 1, 41-62.

[10] G. A. Jones, R. Nedela, M. Škoviera, Regular embeddings of $K_{n,n}$ where n is an odd prime power, submitted.

[11] G. A. Jones, D. Singerman, Belyĭ functions, hypermaps and Galois groups. *Bull. London Math. Soc.* **28** (1996) No. 6, 561-590.

[12] R. Nedela, M. Škoviera, Regular maps from voltage assignments and exponent groups, *European J. Combinat.* **18** (1997), 807-823.

[13] R. Nedela, M. Škoviera, A. Zlatoš, Regular embeddings of complete bipartite graphs, *Discrete Math.* **258** (2002), 379-381.

[14] Ch.-H. Sah, Groups related to compact Riemann surfaces, *Acta Math.* **123** (1969), 13-42.

[15] S. E. Wilson, Cantankerous maps and rotary embeddings of K_n. *J. Combinat. Theory Ser. B* **47** (1989) No. 3, 262–279.