Regular Hamiltonian embeddings of the complete bipartite graph $K_{n,n}$ in an orientable surface

M. J. Grannell, T. S. Griggs
Department of Pure Mathematics
The Open University
Walton Hall
Milton Keynes MK7 6AA
UNITED KINGDOM

M. Knor
Department of Mathematics
Faculty of Civil Engineering
Slovak University of Technology
Radlinského 11
813 68 Bratislava
SLOVAKIA

Abstract

An embedding M of a graph G is said to be regular if and only if for every two triples (v_1, e_1, f_1) and (v_2, e_2, f_2), where e_i is an edge incident with the vertex v_i and the face f_i, there exists an automorphism of M which maps v_1 to v_2, e_1 to e_2 and f_1 to f_2. We show that for $n \not\equiv 0 \pmod 8$ there is, up to isomorphism, precisely one regular Hamiltonian embedding of $K_{n,n}$ in an orientable surface, and that for $n \equiv 0 \pmod 8$ there are precisely two such embeddings. We give explicit constructions for these embeddings as lifts of spherical embeddings of dipoles.

AMS classification: 05C10.

1 Introduction

Topological graph theory is concerned with embedding graphs in surfaces in such a way that the edges of the graph intersect only at the vertices with which they are incident. Such an embedding is called a *map*; see [6] for precise definitions. The surface may be orientable or nonorientable. Amongst the embeddings of a graph G, particular interest arises in those embeddings which possess the greatest possible symmetry. An embedding M of a graph G is said to be *regular* if and only if for every two *flags*, i.e. triples (v_1, e_1, f_1) and (v_2, e_2, f_2), where e_i is an edge incident with the vertex v_i and the face f_i, there exists an automorphism of M which maps v_1 to v_2, e_1 to e_2 and f_1 to f_2. Plainly G can have a regular embedding M only if G is both vertex and edge transitive. Furthermore, the regularity of an embedding M requires that all the face boundaries are of the same length.

We point out that the definition of regularity varies somewhat between authors; see [1] (p.36) for a discussion of the terminology. The definition given here requires the admission of automorphisms which reverse the orientation of the surface in the orientable case. However, many authors require that any global orientation of the surface is preserved and this means that their regular embeddings may be less symmetric.

There is a considerable body of published material relating to regular embeddings. Recent articles include [7] and [10], and the survey papers [9] and [13]. With the exception of complete graphs, see [2, 8], it is perhaps fair to say that there are few definitive results which describe all regular embeddings of a certain type for some particular class of graphs. The purpose of this paper is to determine those regular embeddings of the complete bipartite graph $K_{n,n}$ in an orientable surface whose face boundaries are Hamiltonian cycles. In [3, 12], the authors give one such embedding for each n. In any such embedding M, it is easy to see that $|Aut(M)| = 4n^2$ and that the genus of the surface is $g = (n-1)(n-2)/2$. Computational results given in [4] show the uniqueness of such embeddings for $3 \leq n \leq 7$. We will show that for $n \not\equiv 0 \pmod 8$ there is, up to isomorphism, precisely one such embedding and that for $n \equiv 0 \pmod 8$ there are precisely two such embeddings. We give explicit constructions for these embeddings as lifts of spherical embeddings of dipoles, using current assignments in $\mathbf{Z}_n$. We refer the reader to [6] for a general discussion of voltage graphs.

In a further paper [5], it is shown that for each positive integer n there is a unique regular triangular embedding M^* of the complete tripartite graph $K_{n,n,n}$ in an orientable surface. The two problems are related since, by selecting one of the three sets of the tripartition of $K_{n,n,n}$ and deleting these vertices and the edges incident with them, one may obtain from M^* a regular Hamiltonian embedding of $K_{n,n}$ in an orientable surface. The existence, for $n \equiv 0 \pmod 8$, of a second regular Hamiltonian embedding of $K_{n,n}$ in an orientable surface shows that this process cannot, in general, be reversed.

2 Results

In order to formulate our results it is first necessary to discuss the dipole embedding mentioned in the Introduction. Let M be an embedding in a sphere of a graph with two vertices u and v, and n parallel edges. Then each face of the embedding is a 2-gon. Further, let $a_0, a_1, \ldots, a_{n-1}$ be voltages in the clockwise rotation on the arcs emanating from u, see Figure 1, such that $\{a_0, \ldots, a_{n-1}\} = \{0, \ldots, n-1\}$. Then the voltages around v in the clockwise rotation are $-a_{n-1}, -a_{n-2}, \ldots, -a_0$. Suppose that for each i, $0 \le i \le n-1$, the differences $a_i - a_{i-1}$ are coprime with n (the indices are always taken modulo n).

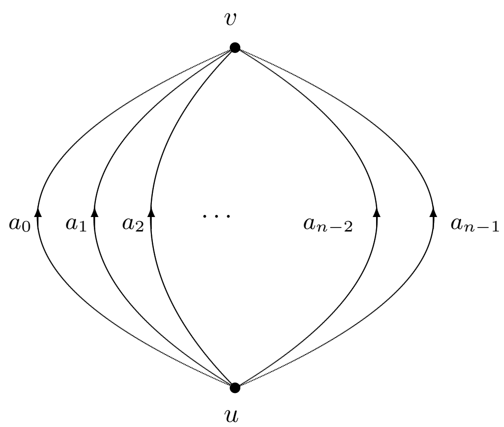

Figure 1: Voltage graph for biembedding.

Now consider the lift of M with voltages over the group $\mathbf{Z}_n$. In the lift we have vertex sets $U = \{u_0, u_1, \ldots, u_{n-1}\}$ and $V = \{v_0, v_1, \ldots, v_{n-1}\}$, and as all $a_i - a_{i-1}$ are coprime with n, each face (2-gon) of M is lifted to a $2n$-gon. Hence, we get an embedding of the complete bipartite graph $K_{n,n}$ in an orientable surface in which every face is bounded by a Hamiltonian cycle. We denote this embedding by $B(u, v; \alpha)$, where α is the permutation $(a_0, a_1, \ldots, a_{n-1})$. The particular case when $\alpha = (0, 1, \ldots, n-1)$ was considered in [11].

Next we examine possible isomorphisms of $B(u, v; \alpha)$ and $B(u, v; \beta)$. Clearly, if $\beta = (a_{k+0}, a_{k+1}, \ldots, a_{k+(n-1)})$, i.e., if β is obtained by *rotating*

α, then the corresponding voltage embeddings (of dipoles) in the sphere are identical. Hence, in this case the lifted embeddings are not only isomorphic, they are identical.

Now suppose that $\beta = (k+a_0, k+a_1, \ldots, k+a_{n-1})$, i.e., β is obtained by *adding* a constant k to every voltage of α, which we denote by $\beta = k + \alpha$. Then a mapping μ, such that $\mu(u_i) = u_i$ and $\mu(v_i) = v_{k+i}$, $0 \le i \le n-1$, is an isomorphism of $B(u, v; \alpha)$ onto $B(u, v; \beta)$.

Next, suppose that $\beta = (ka_0, ka_1, \ldots, ka_{n-1})$, i.e., β is obtained by *multiplying* the voltages of α by a constant k coprime with n, which we denote by $\beta = k\alpha$. Then a mapping μ, such that $\mu(u_i) = u_{ki}$ and $\mu(v_i) = v_{ki}$, $0 \le i \le n-1$, is an isomorphism of $B(u, v; \alpha)$ onto $B(u, v; \beta)$.

Finally, suppose that $\beta = (a_{n-1}, a_{n-2}, \ldots, a_0)$, i.e., β is obtained by *reversing* the permutation α, which we denote by $\beta = \alpha^{-1}$. Then $B(u, v; \beta)$ is isomorphic to $B(u, v; -\beta)$, where $-\beta = (-1)\beta = (-a_{n-1}, -a_{n-2}, \ldots, -a_0)$, so that it is enough to find an isomorphism μ mapping $B(u, v; \alpha)$ onto $B(u, v; -\beta)$. But this can be done by $\mu(u_i) = v_i$ and $\mu(v_i) = u_i$, $0 \le i \le n-1$.

We say that two permutations α and β are *equivalent*, if β can be obtained from α by a sequence of operations consisting of rotating, adding, multiplying and reversing. From the foregoing discussion, it is clear that if two permutations are equivalent, then the corresponding embeddings are isomorphic. In fact the converse is also true.

Lemma 2.1 *If $B(u, v; \alpha)$ and $B(x, y; \beta)$ are isomorphic embeddings then α and β are equivalent permutations.*

Proof First suppose that $B(u, v; \alpha)$ and $B(x, y; \beta)$ are isomorphic embeddings with isomorphism μ sending $\{U\}$ onto $\{X\}$ and $\{V\}$ onto $\{Y\}$.

Assume that $\mu(u_0) = x_{-k_r}$ for some $-k_r \in \mathbf{Z}_n$. In $B(x, y; \beta)$ consider a mapping ν such that $\nu(x_i) = x_{k_r+i}$ and $\nu(y_i) = y_{k_r+i}$, $0 \le i \le n-1$. Then ν is an automorphism of $B(x, y; \beta)$ which maps x_{-k_r} onto x_0. Hence, composing μ with ν we get an isomorphism μ_r mapping $B(u, v; \alpha)$ onto $B(x, y; \beta)$, such that $\mu_r(u_0) = x_0$.

Now assume that $\mu_r(v_0) = y_{-k_a}$. Then $B(x, y; \beta)$ is isomorphic to $B(x, y; k_a + \beta)$, and composing μ_r with this isomorphism we get an isomorphism μ_a mapping $B(u, v; \alpha)$ onto $B(x, y; k_a + \beta)$, such that $\mu_a(u_0) = x_0$ and $\mu_a(v_0) = y_0$.

Further, $\alpha = (\ldots, -d, 0, \ldots)$ for some d coprime with n. Assume that $\mu_a(u_d) = x_{d^*}$. Observe that the 2-gon of the voltage graph with vertices u and v and voltages 0 and $-d$ is lifted to a $2n$-gon with boundary cycle $u_0, v_0, u_d, v_d, \ldots, u_{-d}, v_{-d}$ in $B(u, v; \alpha)$. As μ_a induces an isomorphism of $B(u, v; \alpha)$ onto $B(x, y; k_a + \beta)$, the image of this face is again a face and its boundary cycle is $x_0, y_0, x_{d^*}, y_{q_1}, x_{q_2}, \ldots$. However, as all the faces of

$B(x,y;k_a+\beta)$ are obtained by lifts of 2-gons of a voltage graph, we have $k_a+\beta=(\ldots,-d^*,0,\ldots)$, where d^* is coprime with n. Thus, there is a multiplier k_m such that $k_m d^* = d$. Now $B(x,y;k_a+\beta)$ is isomorphic to $B(x,y;k_m(k_a+\beta))$, and composing μ_a with this isomorphism gives a new isomorphism μ_m mapping $B(u,v;\alpha)$ onto $B(x,y;\beta_m)$, $\beta_m = k_m(k_a+\beta)$, in which $\mu_m(u_0)=x_0$, $\mu_m(v_0)=y_0$, and $\mu_m(u_d)=x_d$. Moreover, the face $f : u_0, v_0, u_d, v_d, \ldots, u_{-d}, v_{-d}$ is mapped onto the face $\mu_m(f) : x_0, y_0, x_d, y_d, \ldots, x_{-d}, y_{-d}$, so that $\mu_m(u_i)=x_i$ and $\mu_m(v_i)=y_i$. Observe that all vertices of the embedded complete bipartite graphs appear on these two faces. It follows that $\alpha=\beta_m$, and, as β_m is equivalent to β, so is α.

Now suppose that $\mu(U)=Y$ and $\mu(V)=X$. As $B(x,y;\beta)$ is identical with $B(y,x;-\beta^{-1})$, the mapping μ takes $B(u,v;\alpha)$ onto $B(y,x;-\beta^{-1})$. Obviously, this reduces the case to the previous one. □

We can now proceed to the statement and proof of our main result.

Theorem 2.1 *If M is a regular Hamiltonian embedding of $K_{n,n}$ in an orientable surface, then M is isomorphic with some $B(u,v;\alpha)$, where $\alpha = \alpha(d) = (0, 1, d+1, d^2+d+1, \ldots, d^{n-2}+d^{n-3}+\ldots+d+1)$ and either*

(i) $d=1$, or

(ii) $n \equiv 0 \pmod 8$ and $d = n/2+1$.

Conversely, if $M = B(u,v;\alpha(d))$, where either (i) or (ii) holds, then M is a regular Hamiltonian embedding of $K_{n,n}$ in an orientable surface.

Proof Let M be a regular Hamiltonian embedding of the complete bipartite graph $K_{n,n}$ in an orientable surface. The embedding M has n faces, each of which contains all $2n$ vertices of the graph. Let f_0 be one of these faces. Denote the vertices of the boundary cycle of f_0 consecutively (say anti-clockwise) by

$$u_0, v_1, u_1, v_2, \ldots, u_{n-1}, v_0.$$

Consider the clockwise rotation around u_0. In this rotation we have a sequence $\ldots, v_0, v_1, v_{d+1}, \ldots$ for some d with $1 \le d \le n-2$. There is an automorphism ψ_k mapping (u_0, u_0v_1, f_0) onto (u_k, u_kv_{k+1}, f_0), $0 \le k \le n-1$. By considering the boundary of f_0, we find that $\psi_k(u_i) = u_{k+i}$ and $\psi_k(v_i) = v_{k+i}$. Hence in the clockwise rotation around u_k we have $\ldots, v_k, v_{k+1}, v_{k+d+1}, \ldots$. There is also an automorphism χ_l mapping (u_0, u_0v_1, f_0) onto (v_l, v_lu_{l-1}, f_0), $0 \le l \le n-1$. By considering the boundary of f_0, we find that $\chi_l(u_i) = v_{l-i}$ and $\chi_l(v_i) = u_{l-i}$. Hence in the clockwise rotation around v_l we have $\ldots, u_{l-(d+1)}, u_{l-1}, u_l, \ldots$. It follows that there is a face f_1 with boundary cycle (reading anti-clockwise)

$$v_1, u_0, v_{d+1}, u_d, v_{2d+1}, u_{2d}, \ldots, u_{-d}.$$

Note that this implies that d is coprime with n.

There is an automorphism θ mapping (u_0, u_0v_1, f_0) onto (u_0, u_0v_{d+1}, f_1). By considering the boundaries of f_0 and f_1 we find that $\theta(u_i) = u_{di}$ and $\theta(v_i) = v_{di+1}$. Then by applying θ^j for $j = 1, 2, \ldots$, it follows that the clockwise rotation around u_0 has the form $\ldots, v_0, v_1, v_{d+1}, v_{d^2+d+1}, \ldots$. Applying ψ_k to this, we deduce that the clockwise rotation around u_k has the form $\ldots, v_k, v_{k+1}, v_{k+d+1}, v_{k+d^2+d+1}, \ldots$, and applying χ_l we see that the <u>clockwise</u> rotation around v_l has the form $\ldots, u_{l-(d^2+d+1)}, u_{l-(d+1)}, u_{l-1}, u_l, \ldots$. Note that these require the n numbers $0, 1, d+1, d^2+d+1, \ldots, d^{n-2}+d^{n-3}+\ldots+d+1$, to be distinct modulo n, and that $d^{n-1}+d^{n-2}+\ldots+d+1 \equiv 0 \pmod{n}$. It follows that $M' = B(u, v; \alpha)$ where $\alpha = (0, 1, d+1, \ldots, d^{n-2}+d^{n-3}+\ldots+d+1)$.

There is an automorphism π mapping (u_0, u_0v_1, f_0) to (u_0, u_0v_1, f_1). By considering the boundaries of f_0 and f_1, we find that $\pi(u_i) = u_{-di}$ and $\pi(v_i) = v_{-di+d+1}$. Applying π to the rotation about u_0 given above, we find that this rotation in the <u>anti-clockwise</u> sense must have the form $\ldots, v_{d+1}, v_1, v_{-d^2+1}, v_{-d^3-d^2+1}, \ldots$. Comparing this with the original version gives, in particular, $v_{-d^2+1} = v_0$ and consequently $d^2 \equiv 1 \pmod{n}$. The equation $d^{n-1} + d^{n-2} + \ldots + d + 1 \equiv 0 \pmod{n}$ implies that $d^n \equiv 1 \pmod{n}$, and so if n is odd we must have $d = 1$.

Now suppose that $d \neq 1$ and consequently that n is even. Then $\{0, 1, d+1, d^2+d+1, \ldots, d^{n-2}+d^{n-3}+\ldots+d+1\} \equiv \{0, 1, d+1, d+2, 2d+2, 2d+3, 3d+3, \ldots, (r-1)d+r\} \pmod{n}$ where $n = 2r$. We require that these n numbers are all distinct modulo n and that $rd + r \equiv 0 \pmod{n}$. It follows that the least value of $l \in \{1, 2, \ldots, n\}$ for which $l(d+1) \equiv 0 \pmod{n}$ should be $l = r$, and the next lowest value should be $l = 2r$. However we also have $(d-1)(d+1) \equiv 0 \pmod{n}$ and $d \in \{1, 2, \ldots, n-1\}$. So we conclude that $d - 1 = r$, giving $d = r + 1$. Since $d^2 = r^2 + 2r + 1 \equiv r^2 + 1 \pmod{n}$, we must have $r^2 \equiv 0 \pmod{n}$. Hence r is even and we may write $n = 4s$ and $d = 2s + 1$.

Now suppose that s is odd; $s = 2t + 1$, say. Then $n = 8t + 4$ and $d = 4t + 3$. We have $(2t+1)(d+1) = (2t+1)(4t+4) = (8t+4)(t+1) \equiv 0 \pmod{n}$, so that $r = 4t+2$ is not the lowest value of l for which $l(d+1) \equiv 0 \pmod{n}$. It follows that if $d \neq 1$ then $n \equiv 0 \pmod{8}$ and $d = n/2+1$. This completes the proof of the first part of the Theorem.

Assume now that $d = 1$, or that $n \equiv 0 \pmod{8}$ and $d = n/2 + 1$. Then d, n are coprime and $d^2 \equiv 1 \pmod{n}$. Furthermore, $\{0, 1, d+1, d^2+d+1, \ldots, d^{n-2}+d^{n-3}+\ldots+d+1\} \equiv \{0, 1, \ldots, n-1\} \pmod{n}$; this is trivial if $d = 1$ while for $n = 8t$ and $d = 4t + 1$ the values on the left-hand side give $\{0, 1, d+1, d+2, 2d+2, 2d+3, 3d+3, \ldots, (r-1)d+r\} \pmod{n}$ where $r = 4t$. For two of these to be congruent modulo n requires the existence of some integer $l \in \{1, 2, \ldots, 4t-1\}$ for which either $l(d+1) \equiv 0$ or $\pm 1 \pmod{n}$. The latter is impossible for any l since $d+1$ and n are even, while the former requires that $l(4t+2) = k(8t)$ for some integer k which is then necessarily a multiple of $2t+1$, giving the minimum positive value of l as $4t$. It follows that $M = B(u, v; \alpha)$ is a Hamiltonian embedding of $K_{n,n}$ in an orientable surface if $\alpha = \alpha(d)$ and conditions (i) or (ii) of the Theorem are satisfied. It remains to prove that the embedding is regular. To do this we now define mappings of the vertices, show that these are automorphisms of the embedding, and establish that every flag may be mapped to every other flag by a suitable combination of these mappings.

The mappings $\psi_k, \chi_k, \theta, \pi,\ 0 \le k \le n-1$, are defined as follows:

$$\begin{array}{rclrcl} \psi_k(u_i) & = & u_{k+i}, & \psi_k(v_i) & = & v_{k+i}, \\ \chi_k(u_i) & = & v_{k-i}, & \chi_k(v_i) & = & u_{k-i}, \\ \theta(u_i) & = & u_{di}, & \theta(v_i) & = & v_{di+1}, \\ \pi(u_i) & = & u_{-di}, & \pi(v_i) & = & v_{-di+d+1}. \end{array}$$

We also require the rotations at u_i and v_i, $0 \le i \le n-1$, which may be obtained from Figure 1 as

$$\begin{array}{ll} u_i: & v_i, v_{i+1}, v_{i+d+1}, v_{i+d^2+d+1}, \ldots, v_{i+d^{n-2}+d^{n-3}+\ldots+d+1} \\ v_i: & u_{i-(d^{n-2}+d^{n-3}+\ldots+d+1)}, \ldots, u_{i-(d^2+d+1)}, u_{i-(d+1)}, u_{i-1}, u_i \end{array}$$

The algebraic form of these may be simplified using the identity $d^2 \equiv 1 \pmod{n}$. It is a routine matter to check that the application of each of the mappings $\psi_k, \chi_k, \theta, \pi$ to these rotations gives other rotations in $B(u, v; \alpha)$, so that these mappings are indeed automorphisms of the embedding. For example, applying π to the rotation at u_i gives the rotation at u_{-di} as

$$u_{-di}: v_{-di+d+1}, v_{-di+1}, v_{-di-d^2+1}, v_{-di-d^3-d^2+1}, \ldots, v_{-di-d^{n-1}-d^{n-2}-\ldots-d^2+1}$$

Replacing $-di$ by j and using $d^2 \equiv 1 \pmod{n}$, this may be written as

$$u_j: v_{j+d+1}, v_{j+1}, v_j, v_{j-d}, \ldots, v_{j+d+2}$$

which is the same as that obtained from Figure 1 with the orientation reversed. In fact, ψ_k and θ preserve orientation while χ_k and π reverse orientation.

To prove that the embedding is regular, select integers $k, l \in \{0, 1, \ldots, n-1\}$. We prove first that there is an automorphism sending the flag (u_0, u_0v_1, f_0) to (u_k, u_kv_l, f^*) where f_0 and f^* are the faces on the left hand side of the arcs u_0v_1 and u_kv_l respectively. To do this, take j such that $d^j + d^{j-1} + \ldots + d + 1 \equiv l - k \pmod{n}$. Then $\psi_k\theta^j(u_0) = u_k$ and $\psi_k\theta^j(v_1) = v_{d^j+d^{j-1}+\ldots+d+1+k} = v_l$, and it follows that $\psi_k\theta^j$ is the required automorphism. Furthermore, $\psi_k\theta^j\pi$ sends the flag (u_0, u_0v_1, f_0) to (u_k, u_kv_l, f') where f' is the face on the right hand side of the arc u_kv_l.

Next we prove that there is an automorphism sending the flag (u_0, u_0v_1, f_0) to (v_l, v_lu_k, f^*). With j as in the previous paragraph, $\chi_l\theta^j(u_0) = v_l$ and $\chi_l\theta^j(v_1) = u_{l-(d^j+d^{j-1}+\ldots+d+1)} = u_k$, and it follows that $\chi_l\theta^j$ is the required automorphism. Furthermore, $\chi_l\theta^j\pi$ sends the flag (u_0, u_0v_1, f_0) to (v_l, v_lu_k, f').

We have shown that the flag (u_0, u_0v_1, f_0) may be mapped to any other flag by an automorphism of the embedding. Consequently any flag may be mapped to any other flag and the embedding is regular. □

Corollary 2.1.1 *Up to isomorphism, for* $n \not\equiv 0$ *(mod* 8*)* $B(u, v; \alpha(1))$ *is the unique regular Hamiltonian embedding of* $K_{n,n}$ *in an orientable surface. For* $n \equiv 0$ *(mod* 8*) there are precisely two nonisomorphic embeddings of this type, namely* $B(u, v; \alpha(d))$ *for* $d = 1$ *and* $d = n/2 + 1$*.*

Proof The result follows from the Theorem once it is shown that in the $n \equiv 0 \pmod 8$ case, the two embeddings are nonisomorphic. This can be established using Lemma 2.1. Any permutation $\beta = (b_0, b_1, \ldots, b_{n-1})$ equivalent to $\alpha(1)$ will be obtained from $\alpha(1)$ by a finite sequence of the operations multiplication, reversal, rotation and addition. It follows that $b_{i+1} - b_i \equiv b_{j+1} - b_j \pmod n$ for any i, j. However, the permutation $\alpha(n/2 + 1)$ does not have this property and consequently is not equivalent to $\alpha(1)$. □

Acknowledgement

This work was done during a visit by the third author to the Open University. He thanks the University for its hospitality and all authors thank the Leverhulme Trust for financial support under grant F/00269/E.

References

[1] D. Archdeacon, Topological graph theory - a survey, *Congr. Numer.* **115** (1996), 5-54.

[2] N. L. Biggs, Classification of complete maps on orientable surfaces, *Rend. Mat.* **VI 4** (1971), 645-655.

[3] N. L. Biggs and A. T. White, "Permutation groups and combinatorial structures", Cambridge University Press, Cambridge, 1979.

[4] M. J. Grannell, T. S. Griggs and M. Knor, Biembeddings of Latin squares and Hamiltonian decompositions, *submitted.*

[5] M. J. Grannell, T. S. Griggs, M. Knor and J. Širáň, Triangulations of orientable surfaces by complete tripartite graphs, *submitted.*

[6] J. L. Gross and T. W. Tucker, “Topological Graph Theory”, John Wiley, New York, 1987.

[7] R. Jajcay, On a construction of infinite families of regular Cayley maps, *Combinatorica* **18** (1998), no. 2, 191-199.

[8] L. D. James and G. A. Jones, Regular orientable imbeddings of complete graphs, *J. Combin. Theory B* **39** (1985), 353-367.

[9] R. Nedela, Regular maps - combinatorial objects relating different fields of mathematics, *J. Korean Math. Soc.* **38** (2001), no. 5, 1069-1105.

[10] R. Nedela and M. Škoviera, Regular maps from voltage assignments and exponent groups, *European J. Combin.* **18** (1997), no. 7, 807-823.

[11] S. Stahl and A. T. White, Genus embeddings for some complete tripartite graphs, *Discrete Math.* **14** (1976), 279-296.

[12] A. T. White, Symmetrical maps, *Congr. Numer.* **45** (1984), 257-266.

[13] A. T. White, Cayley maps, *Congr. Numer.* **115** (1996), 105-130.