INDEPENDENCE NUMBER IN PATH GRAPHS

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Abstract. In the paper we present results, which allow us to compute the independence numbers of P_2-path graphs and P_3-path graphs of special graphs. As $P_2(G)$ and $P_3(G)$ are subgraphs of iterated line graphs $L^2(G)$ and $L^3(G)$, respectively, we compare our results with the independence numbers of corresponding iterated line graphs.

1. Introduction

Let G be a graph, $k \geq 1$, and let $\mathcal{P}_k$ be the set of all paths of length k (i.e., with $k{+}1$ vertices) in G. The vertex set of a **path graph** $P_k(G)$ is the set $\mathcal{P}_k$. Two vertices of $P_k(G)$ are joined by an edge if and only if the edges in the intersection of the corresponding paths form a path of length $k{-}1$, and their union forms either a cycle or a path of length $k{+}1$. This means that the vertices are adjacent if and only if one can be obtained from the other by "shifting" the corresponding paths in G.

Path graphs were investigated by Broersma and Hoede in [4] as a natural generalization of line graphs, since $P_1(G)$ is the line graph $L(G)$ of G. However, there is also another connection between path graphs and line graphs. Denote by $L^k(G)$ the graph, obtained from G by applying k times the line graph operator. Then $P_k(G)$ is a subgraph of $L^k(G)$, and $P_2(G)$ is a spanning subgraph of $L^2(G)$.

Traversability of P_2-path graphs is studied in [16], and a characterization of P_2-path graphs is given in [4] and [13]. Distance properties of path graphs are studied

1991 *Mathematics Subject Classification.* 05C69, 05C38.

Key words and phrases. Path graph, independence number, iterated line graph, graph dynamics, (non)deterministic algorithm.

[1)] Supported by VEGA grant 1/9176/02

[2)] Supported by Kuwait University grant #SM 02/00

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in [3], [8] and [9], and [10], [5] and [2] are devoted to connectivity of path graphs. Isomorphism of path graphs is studied in [1] and [14].

Let G be a graph. By $V(G)$ and $E(G)$ we denote the vertex set and edge set of G, respectively, and for each vertex v of G, by $deg_G(v)$ we denote the degree of v in G. A subset A of the set of vertices of G is **independent**, if there is no edge in G connecting a pair of vertices of A. The maximum size of the set of independent vertices of G is the **independence number** $\beta(G)$.

In this paper we study the independence number of P_k-path graphs. Since the problem is complicated in general, we restrict ourselves to cases $k = 2$ and $k = 3$. We construct maximum sets of independent vertices of $P_k(G)$ for special classes of graphs G, $2 \leq k \leq 3$, and we compare the results with the independence numbers of $L^k(G)$.

The outline of the paper is as follows. In part 2 we study iterated line graphs, while parts 3 and 4 are devoted to P_2-path graphs and P_3-path graphs, respectively.

2. Iterated line graphs

Let G be a graph and let $L^2(G)$ be its second iterated line graph. It is easy to see that there is a one-to-one correspondence between the vertices of $L^2(G)$ and the paths of length two in G, see [15]. For that reason, we will identify these two objects.

Two distinct vertices are adjacent in $L^2(G)$ if and only if the corresponding paths share an edge in common. Hence, the set of vertices in $L^2(G)$ is independent if and only if the corresponding paths of length two are edge-disjoint in G.

In the pioneering paper [11] there is the following lemma:

Lemma 1. *Let G be a connected graph with even number of edges. Then the edge set of G can be decomposed into edge-disjoint paths of length two.*

As the paper [11] is in Slovak, it is not familiar in general. For that reason we include a sketch of the proof from [11].

Sketch of the proof of Lemma 1. Let us direct all the edges of G. This orientation can be done in such a way, that to every vertex of G there is directed an even number of arcs. (If this is not the case, than denote by P an undirected path joining two vertices of odd in-degree and reverse the orientation of all arcs of P. Proceed in this process until no vertex of odd in-degree remains.)

Finally, for every vertex v of G, partition the arcs terminating in v into pairs, to form edge-disjoint paths of length two. $\square$

As a straightforward consequence of Lemma 1 we have the following statement:

Theorem 2. *For every connected graph G we have $\beta(L^2(G)) = \left\lfloor \frac{|E(G)|}{2} \right\rfloor$.*

It is well-known that for general graph G the problem of finding its independence number is NP-hard, see [6]. If we consider line graphs, maximum independent sets in $L(G)$ correspond to maximum matchings in G. Hence, the problem of finding the independence number of $L(G)$ is polynomial, see [12]. And as we have already shown, the problem of finding the independence number of $L^2(G)$ is trivial.

In [15] we present bounds for the numbers of vertices in iterated line graphs. As a consequence, for r-regular graphs, $r \geq 2$, and $k \geq 2$ we have

$$\beta(L^k(G)) = \left\lfloor \frac{n}{2} \cdot \prod_{i=0}^{k-1} \left[2^{i-1} \cdot (r-2) + 1\right] \right\rfloor ,$$

where n is the number of vertices of G. Moreover, for every graph G distinct from a path, a cycle and a claw, there are constants c_1 and c_2, $c_1 > 0$ and $c_2 \geq 1$, such that

$$\beta(L^k(G)) > c_1 \cdot 2^{\binom{k-2}{2}} \cdot c_2^{k-1}$$

for every $k \geq 0$.

3. P_2-PATH GRAPHS

We present a non-deterministic algorithm for finding a maximum independent set of $P_2(G)$. This algorithm is not important from the algorithmic point of view, however, it shows the structure of any maximum independent set, see Theorem 3.

Algorithm 1. Let G be a graph with the maximum degree $\Delta(G)$.

Step 0. Set $G' = G$, $d = \Delta(G)$ and $I = \emptyset$.

Step 1. Choose a vertex v of G', such that $2 \leq \deg_{G'}(v) \leq d$. Add to I all paths of G' of the form (x, v, y) where both xv and vy are edges of G'. Set $d = deg_{G'}(v)$. Now delete from G' the vertex v and all edges incident to v, and denote the resulting graph by G'.

Step 2. If $\Delta(G') \geq 2$ then go to Step 1, otherwise terminate the algorithm and give the set I on the output.

Theorem 3. *Let I be any maximum independent set of vertices of $P_2(G)$, i.e., $|I| = \beta(P_2(G))$. Then, choosing the vertices in a clever way, I can be found by Algorithm 1.*

Proof. First, observe that if we proceed in Algorithm 1 in any way, we construct an independent set of vertices of $P_2(G)$. The reason is that if two paths of I share an edge, say uv, then these paths are (u, v, x) and (u, v, y) for suitable x and y.

Let I be a maximum independent set of $P_2(G)$. For every vertex u of G, denote by I_u the paths of I which have u in the center. Further, denote by T_u the set of endvertices of paths in I_u. As I is a maximum independent set, for every $x, y \in T_u$, $x \neq y$, we have $(x, v, y) \in I_u$. Hence, $|I_u| = \binom{|T_u|}{2}$

Let v be a vertex of G such that $|I_v|$ is maximum. To prove the statement, it is enough to show that $|T_v| = deg_G(v)$, i.e., that all neighbours of v occur in T_v.

Suppose that there is a neighbour w of v, such that $w \notin T_v$. If the edge vw does not appear in the paths of I, we can add to I all paths of the form (w, v, x), $x \in T_v$, to obtain a larger independent set, which contradicts the maximality of I. Hence, we may assume that $v \in T_w$. By the choice of v, we have $|I_v| \geq |I_w|$, and consequently $|T_v| \geq |T_w|$. Remove from I all paths of the form (x, w, v), add the paths (w, v, y), and denote the resulting set by I'. Obviously, I' forms an independent set of vertices in $P_2(G)$, and

$$|I'| = |I| - \big(|T_w| - 1\big) + |T_v| > |I|,$$

a contradiction. $\square$

Now we focus our attention to some classes of graphs and determine the independence numbers of their P_2-path graphs.

As a complete graph K_n is vertex-transitive and deleting a vertex from K_n we receive a K_{n-1}, we can choose the vertices for Algorithm 1 in arbitrary order. Thus, we have

$$\beta(P_2(K_n)) = \binom{n-1}{2} + \binom{n-2}{2} + \cdots + \binom{2}{2} = \binom{n}{3}.$$

Observe that $\beta(L^2(K_n)) = \lfloor \frac{1}{2}\binom{n}{2} \rfloor$, $|V(L^2(K_n))| = |V(P_2(K_n))| = 3\binom{n}{3}$, the graph $L^2(K_n)$ is regular of degree $4n-10$ and $P_2(K_n)$ is regular of degree $2n-4$.

For complete bipartite graphs $K_{n,n}$ the situation is more complicated, because there are many ways for choosing the order of vertices for our algorithm. In fact, here the algorithm is useless. However, we can determine $\beta(P_2(K_{n,n}))$ in other way.

Since $K_{n,n}$ is a bipartite graph, $P_2(K_{n,n})$ is bipartite as well. Denote by $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ the bipartition of the vertex set of $K_{n,n}$. The vertices of $P_2(K_{n,n})$ can be partitioned into four-cycles and six-cycles

$$V(P_2(K_{n,n})) = \bigcup_{a \neq b} \{(u_a, v_a, u_b), (v_a, u_b, v_b), (u_b, v_b, u_a), (v_b, u_a, v_a)\} \bigcup$$

$$\bigcup_{a \neq b \neq c \neq a} \{(u_a, v_b, u_c), (v_b, u_c, v_a), (u_c, v_a, u_b), (v_a, u_b, v_c), (u_b, v_c, u_a), (v_c, u_a, v_b)\}.$$

Hence, $P_2(K_{n,n})$ is a bipartite graph having a linear factor. Thus, $\beta(P_2(K_{n,n})) = \frac{|V(P_2(K_{n,n}))|}{2} = \frac{n^3-n^2}{2}$. Observe that $\beta(L^2(K_{n,n})) = \lfloor \frac{n^2}{2} \rfloor$, the graph $L^2(K_{n,n})$ is regular of degree $4n-6$ and $P_2(K_{n,n})$ is regular of degree $2n-2$.

Using Algorithm 1, the maximum independent set of $P_2(K_{n,n})$ can be obtained by choosing, in each step, the vertex of the maximum degree. This gives an independent set of size $n \cdot \binom{n}{2} = \frac{n^3-n^2}{2} = \beta(P_2(K_{n,n}))$. Analogous situation has appeared in the trivial case of K_n. Unfortunately, there are graphs for which we cannot start with a vertex of maximum degree.

Let us denote by $T_{t,s}$ a tree, obtained from t vertex disjoint stars $K_{1,s-1}$ and one extra vertex which is joined to all central vertices of the stars. We determine $\beta(P_2(T_{t,s}))$ using Algorithm 1. In $T_{t,s}$ there are t vertices of degree s, one vertex of degree t and some vertices of degree 1. As $T_{t,s}$ is symmetric, in Algorithm 1 the size of I is determined by the order in which it chooses the central vertex. Hence, denote by $\beta(k)$ the size of independent set obtained by choosing k vertices of degree s, then the central vertex of degree $t-k$, and finishing with the remaining $t-k$ vertices of degree $s-1$, $0 \leq k \leq t$. This gives

$$\beta(k) = k\binom{s}{2} + \binom{t-k}{2} + (t-k)\binom{s-1}{2} = \frac{1}{2}\Big[k^2 + k(2s-2t-1) + c\Big],$$

where c is a constant depending only on t and s. Hence, $\beta(k)$ is maximum if $k=0$ or $k=t$. Since $\beta(t) > \beta(0)$ is equivalent to $t^2 + t(2s-2t-1) > 0$, which gives $2s-1 > t$, we cannot start with the vertex of maximum degree if $2s-1 > t > s$. In fact, this shows that the problem of finding the independence number of $P_2(G)$ is not trivial in general.

4. P_3-PATH GRAPHS

As the problem is complicated for P_3-path graphs in general, we consider only graphs G of girth at least 4. We start with a non-deterministic algorithm, which is similar to Algorithm 1.

Algorithm 2. Let G be a graph of girth at least 4 and the maximum degree $\Delta(G)$.

Step 0. Set $G' = G$, $d = \Delta(G)$ and $I = \emptyset$.

Step 1. Choose an edge uv of G', such that $2 \le deg_{G'}(u) \le d$. Add to I all paths of G' of the form (x, u, v, y) where both xu and vy are edges of G'. Set $d = deg_{G'}(u)$. Now delete from G' the edge uv, and denote the resulting graph by G'.

Step 2. If there are paths of length 3 in G then go to Step 1, otherwise terminate the algorithm and give the set I on the output.

Theorem 4. *Let G be a graph of girth at least 4. Then choosing the vertices in a clever way, Algorithm 2 finds a maximum independent set of $P_3(G)$.*

Proof. First, observe that Algorithm 2 constructs an independent set of vertices of $P_3(G)$. The reason is that if uv is a central edge in a path in I, then it does not occur as an endedge of any path in I.

Let I be a maximum independent set of $P_3(G)$. For every edge uv of G, denote by I_{uv} the paths of I which have the edge uv in the middle. Further, let

$$T_u^{uv} = T_u^{vu} = \{x;\ (x, u, v, y) \in I_{uv} \text{ for suitable } y\}.$$

As I is a maximum independent set, for every $x \in T_u^{uv}$ and $y \in T_v^{uv}$, we have $(x, u, v, y) \in I$. (Observe that G has no triangles.)

Let us consider an arbitrarily chosen vertex v of G. Among all neighbours of v, let u have the maximum value of $|T_u^{vu}|$. Suppose that $|T_u^{vu}| > 0$. Moreover, suppose that w is a neighbour of v, $w \ne u$, such that $w \notin T_v^{uv}$. If (u, v, w) does not occur as a subpath in elements of I, then we can add to I all paths (x, u, v, w), $x \in T_u^{vu}$, to obtain a larger independent set. Hence, (u, v, w) is a subpath in an element of I. Since $w \notin T_v^{uv}$, it follows that $u \in T_v^{wv}$. By the choice of u, $|T_u^{vu}| \ge |T_w^{vw}|$. Remove from I all paths of the form (u, v, w, y), $y \in T_w^{vw}$, add the paths (x, u, v, w), $x \in T_u^{vu}$, and denote the resulting set by I'. Obviously, I' forms an independent set in $P_3(G)$ and

$$|I'| = |I| - |T_w^{vw}| + |T_u^{vu}| \ge |I|.$$

Observe that our change of I to I' is local. Namely, if we determine the sets T_x^{yx} for I', these sets will coincide with those determined for I, except for the case $x = v$. If we denote $r = deg_G(v)$, then continuing in these local changes for other neighbours of v, we get a maximum independent set with $|T_v^{uv}| = r - 1$.

Now we repeat this process with an edge zv of G, such that $|T_z^{vz}|$ is maximum among the neighbours of v, $z \ne u$. Again, suppose that $|I_{vz}| > 0$. We must avoid the edge uv, but after the local changes all the remaining edges will occur in I_{vz}. We get $|T_v^{zv}| = r - 2$. Continuing in these changes with other edges we see that the neighbours of v can be labelled by $u_1, u_2, \ldots, u_r$, so that either $|T_v^{vu_i}| = r - i$, or $|T_v^{vu_i}| = 0$ (if $|I_{vu_i}| = 0$). Obviously, if $|T_v^{vu_i}| = 0$ then $|T_v^{vu_j}| = 0$ for every $j > i$.

Assume that these local changes were done for all vertices v of G. Among all edges with $|I_{ab}| > 0$ let ab have maximum value of $deg_G(a)$. Moreover, let ab be the edge for which $|T_a^{ba}| = deg_G(a) - 1$.

If there is a neighbour c of b, $c \neq a$, such that $c \notin T_b^{ab}$, then remove from I all paths of the form (a, b, c, y), $y \in T_c^{bc}$, add the paths (x, a, b, c), $x \in T_a^{ba}$, and denote the resulting set by I'. Obviously, I' forms an independent set in $P_3(G)$ and

$$|I'| = |I| - |T_c^{bc}| + |T_a^{ba}| \geq |I|.$$

We can do these changes with other neighbours of b as well, to obtain $|T_b^{ab}| = deg_G(b) - 1$. As we already assumed $|T_a^{ba}| = deg_G(a) - 1$, the result follows. □

Observe that we did not prove that every maximum independent set of P_3-path graph can be found by Algorithm 2. The problem is that the inequalities are not sharp in the previous proof.

Now we focus our attention to graphs which are decomposable into r linear factors, $r \geq 2$. The following (deterministic) algorithm finds a maximum independent set of vertices of $P_3(G)$.

Algorithm 3. Let G be a graph of girth at least 4, and let $F_1, F_2, \ldots, F_r$ be the factorization of G.

Step 0. Set $G' = G$, $I = \emptyset$ and $i = 1$.

Step 1. For each edge uv of F_i, add to I all paths of G' of the form (x, u, v, y) where both xu and vy are edges of G'. Then delete from G' all edges of F_i and denote the resulting graph by G'.

Step 2. If $i < r - 1$ then increase i by 1 and go to Step 1, otherwise terminate the algorithm and give the set I on the output.

Theorem 5. *Let G be a graph of girth at least 4 which is decomposable into r linear factors, $r \geq 2$. Then Algorithm 3 finds a maximum independent set of $P_3(G)$, and the size of this set is $\frac{n}{12}[2r^3 - 3r^2 + r]$ where n is the number of vertices of G.*

Proof. As Algorithm 3 is just a special case of Algorithm 2, it constructs an independent set of vertices of $P_3(G)$. Moreover, the size of this set is $\frac{n}{2}(r-1)^2 + \frac{n}{2}(r-2)^2 + \cdots + \frac{n}{2}1^2 = \frac{n}{12}[2r^3 - 3r^2 + r]$.

As we have already shown in the proof of Theorem 4, in $P_3(G)$ there is a maximum independent set I, such that for every vertex v its neighbours can be labelled by $u_1, u_2, \ldots, u_r$, so that either $|T_v^{vu_i}| = r - i$, or $|T_v^{vu_i}| = 0$. (We use the notation from the previous proof.) Thus,

$$|I| = \sum_{uv \in E(G)} |T_u^{uv}| \cdot |T_v^{uv}|,$$

where there are at most n multipliers of size $r-i$, $1 \leq i \leq r-1$. Since $ab+ac < a^2+bc$ if $a > b \geq c \geq 0$, we have $|I| \leq \frac{n}{2}(r-1)^2 + \frac{n}{2}(r-2)^2 + \cdots + \frac{n}{2}1^2 = \frac{n}{12}[2r^3 - 3r^2 + r]$. However, Algorithm 3 finds an independent set which has exactly this size, and hence, it finds a maximum independent set in $P_3(G)$. □

By Theorem 5, for complete bipartite graph $K_{n,n}$ we have $\beta(P_3(K_{n,n})) = \frac{2n}{12}[2n^3 - 3n^2 + n]$. Observe that $P_3(K_{n,n})$ has $n^2(n-1)^2 = n^4 - 2n^3 + n^2$ vertices and degree $2n - 4$, but it is not bipartite. The odd girth of $P_3(K_{n,n})$ is 9. On the other hand, $L^3(K_{n,n})$ has $2n^4 - 5n^3 + 3n^2$ vertices and degree $8n - 14$, and $\beta(L^3(K_{n,n})) = \lfloor \frac{n^3-n^2}{2} \rfloor$.

Analogously, $\beta(P_3(Q_r)) = \frac{2^r}{12}[2r^3 - 3r^2 + r]$ if Q_r denotes an r-dimensional cube. The graph $P_3(Q_r)$ has $2^{r-1}(r^3 - 2r^2 + r)$ vertices and degree $2r - 2$. On the other hand, $L^3(Q_r)$ has $2^{r-1}(2r^3 - 5r^2 + 3r)$ vertices and degree $8r - 14$, and $\beta(L^3(Q_r) = \lfloor\frac{2^{r-1}(r^2-r)}{2}\rfloor$.

We remark that in a sense, the results of section 3 can be generalized for P_{2t}-path graphs and the results of section 4 can be generalized for P_{2t+1}-path graphs, $t \geq 2$. Hovewer, in the generalized algorithms we cannot simply delete edges from G, but we must forbid these edges as "next by central" edges of paths in I. Moreover, there are problems with graphs of "small" girths. So we can either work with graphs of "large" girths (for P_s-path graphs we require the girth at least $s + 1$), or we can turn our attention to walk graphs instead of the path graphs, see [7].

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