TRIANGULATIONS OF ORIENTABLE SURFACES BY COMPLETE TRIPARTITE GRAPHS

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Abstract. Orientable triangular embeddings of the complete tripartite graph $K_{n,n,n}$ correspond to biembeddings of Latin squares. We show that if n is prime there are at least $e^{n \ln n - n(1+o(1))}$ nonisomorphic biembeddings of cyclic Latin squares of order n. If $n = kp$, where p is a large prime number, then the number of nonisomorphic biembeddings of cyclic Latin squares of order n is at least $e^{p \ln p - p(1+\ln k + o(1))}$. Moreover, we prove that for every n there is a unique regular triangular embedding of $K_{n,n,n}$ in an orientable surface.

1. Introduction

For more than a century, graphs have been studied as objects embedded in surfaces. A prominent example is the problem of determining the smallest genus of an orientable surface that embeds K_n, the complete graph of order n. In its dual form this is the famous Heawood map colouring problem. It took more than three quarters of a century until a complete solution (with extension to nonorientable surfaces) appeared [7], using methods that gave birth to modern topological graph theory [6].

A natural design-theoretic interpretation of genus embeddings of K_n arises when the embeddings are triangular. This happens in the orientable case if and only if $n \equiv 0, 3, 4$ or $7 \pmod{12}$ and in the nonorientable case if and only if $n \equiv 0$ or $1 \pmod 3$, $n \neq 3, 4$ or 7, see [7,9]. In the latter case the triangular faces form a *twofold triple system*, TTS(n), and in the former case a *Mendelsohn triple system*, MTS(n). See [2] for definitions of design theoretic terms. We may thus say that the designs are *embedded* in the surface. The problem of determining precisely which TTS(n) and MTS(n) can be so embedded was solved by Ducrocq and Sterboul, [3].

Key words and phrases. Latin square, triangulation, orientable embedding, regular embedding.

If n is odd, it may be possible to colour the faces of a triangular embedding of K_n using two colours, say black and white, in such a way that adjacent faces receive different colours. Such embeddings are said to be *face 2-colourable* and the faces of each colour class form a *Steiner triple system*, STS(n). Again, we say that each $STS(n)$ is embedded in the surface and that the pair of STS(n)s is *biembedded*.

Trying to go beyond Steiner, Mendelsohn and twofold triple systems one may ask about possibilities of representing other designs and combinatorial structures on surfaces. For block designs such a study has been initiated in [10] where surface realizations of designs are obtained via embeddings of the associated bipartite point-block graphs. In the case of embeddings of Steiner triple systems this approach corresponds to face 2-colourable embeddings of complete graphs whose faces in one of the two colour classes correspond to the Steiner triple system.

The general approach of applying design theoretic methods to problems of topological graph theory has been very profitable. For many years after the publication of [7], only a small number of triangulations of K_n were known for any n. In [1] and [5] recursive constructions were described to establish, for values of n lying in certain residue classes, the existence of $2^{cn^2-o(n^2)}$ nonisomorphic face 2-colourable triangulations of K_n using Steiner triple systems of order n.

In this paper, we turn our attention to the study of triangulations of the complete tripartite graph $K_{n,n,n}$ and in particular to those triangulations which are face 2-colourable. One motivation for this is that in [5], the constructions presented use "bridges" which are face 2-colourable triangulations of $K_{n,n,n}$. The second reason is that such embeddings correspond to biembeddings of two Latin squares. Also in [5] it was proved that for $n \equiv 0 \pmod 3$ there are at least $2^{n^2/9-o(n^2)}$ nonisomorphic biembeddings of this type. The Latin squares involved in these constructions are not identified. In this paper we use a lift of the embedding of a dipole with multiple edges in a sphere, see also [8], to obtain $e^{n \ln n - n(1+o(1))}$ nonisomorphic biembeddings of cyclic Latin squares in the case where n is a prime. We remark that, as our embeddings are obtained by voltage assignments on $\mathbb{Z}_n$, the automorphism group of each embedding has order at least n.

In a previous paper, [4], it was shown that face 2-colourability of triangulations of $K_{n,n,n}$ is equivalent to orientability. The same paper also contains enumeration results for biembeddings of Latin squares of order 7 or less. Relatively few of these biembeddings have a "large" automorphism group. For example among the 23,664 nonisomorphic biembeddings of Latin squares of order 7 given in [4] there are only 18 with an automorphism group of order at least 7. Of these, 13 can be obtained by the dipole construction. We also find that for each n, $2 \leq n \leq 7$, there is only one regular triangular embedding of $K_{n,n,n}$ in an orientable surface. Motivated by this, we prove that for any n, $n \geq 2$, there is a unique regular orientable triangular embedding of $K_{n,n,n}$.

2. Results

For triangulations of $K_{n,n,n}$ in an orientable surface we use lifts and voltage assignments. Readers unfamiliar with these terms are directed to [6]. Let M be an embedding in a sphere of a graph with two vertices u and v and n parallel edges. Then each face of the embedding is a 2-gon. Further, let $a_0, a_1, \ldots, a_{n-1}$ be voltages in the clockwise rotation on the arcs emanating from u, see Figure 1, such that $\{a_0, \ldots, a_{n-1}\} = \{0, \ldots, n-1\}$. Then the voltages around v in the clockwise rotation are $-a_{n-1}, -a_{n-2}, \ldots, -a_0$. Suppose that for each i, $0 \leq i \leq n-1$, the differences $a_i - a_{i-1}$ are coprime with n (the indices are always taken modulo n).

Now consider the lift of M with voltages over the group $\mathbb{Z}_n$. In the lift we have vertex sets $U = \{u_0, u_1, \ldots, u_{n-1}\}$ and $V = \{v_0, v_1, \ldots, v_{n-1}\}$, and as all $a_i - a_{i-1}$ are coprime with n, each face (2-gon) of M is lifted to a $2n$-gon. Hence, we get an embedding of the complete bipartite graph $K_{n,n}$ in an orientable surface in which every face is bounded by a Hamiltonian cycle. We denote this embedding by $B(u, v; \alpha)$, where α is the permutation $(a_0, a_1, \ldots, a_{n-1})$. Now place into each of the n faces of $B(u, v; \alpha)$ a vertex and join it to all the vertices lying on the boundary of the face, to create a triangular embedding of the complete tripartite graph $K_{n,n,n}$ in an orientable surface. In what follows, we denote this embedding by $T(u, v; \alpha)$. We remark that $T(u, v; \alpha)$ has $3n$ vertices, $3n^2$ edges and $2n^2$ faces, and so Euler's formula gives the genus of this embedding as $(n-1)(n-2)/2$.

M :

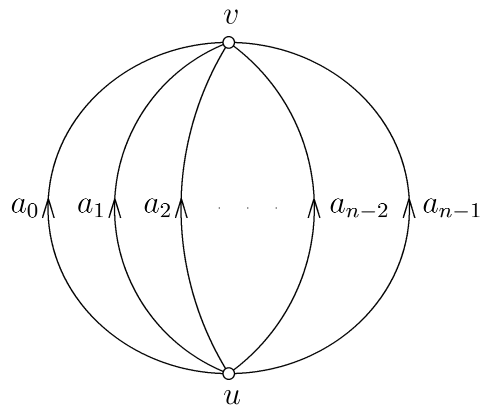

Figure 1

By the construction, $T(u, v; \alpha)$ is an orientable triangulation. It is also face 2-colourable. To show this, let $W = \{w_0, w_1, \ldots, w_{n-1}\}$ be the set of vertices of $T(u, v; \alpha)$ disjoint from U and V. Then, reading clockwise, some of the boundary cycles of triangles are $u_a v_b w_c$, for suitable a, b and c, while the others are $u_a w_c v_b$. As each edge of the embedding is incident to one triangle of each form, the embedding is face 2-colourable.

Now choose one colour class of a $T(u, v; \alpha)$. Then for each pair of vertices $u_i \in U$ and $v_j \in V$ there is a unique $w_k \in W$ such that the vertices u_i, v_j and w_k form a triangle in the colour class. Hence, each colour class forms a *transversal design*, $TD(3, n)$. Recall that a $TD(3, n)$ can be represented by a Latin square in such a way that for every triple u_i, v_j and w_k in the design we put the number k into the i-th row and j-th column of an $n \times n$ square table. As each value w_k appears in each row u_i exactly once, and analogously it appears exactly once in each column v_j, the table is a Latin square. Thus, face 2-colourable embeddings of $K_{n,n,n}$ correspond to biembeddings of Latin squares.

Suppose that w_i is placed into that face of $B(u, v; \alpha)$, which is obtained by lifting the 2-gon with voltages a_i and $-a_{i-1}$, $0 \leq i \leq n-1$. Then in one colour class of $T(u, v; \alpha)$ we have triangles $u_j v_{j+a_0} w_0$, $u_j v_{j+a_1} w_1$, $\ldots$, $u_j v_{j+a_{n-1}} w_{n-1}$, where $0 \leq j \leq n-1$, and in the other class we have $v_j u_{j-a_{n-1}} w_0$, $v_j u_{j-a_0} w_1$, $\ldots$, $v_j u_{j-a_{n-2}} w_{n-1}$. Hence, each of these two Latin squares has constant diagonals running from "top left" to "bottom right", so that both these squares are cyclic. We prove here:

Theorem 1. *If n is prime then there are at least $\frac{n!}{6n^2(n-1)}$ nonisomorphic embeddings $T(u,v;\alpha)$ of the complete tripartite graph $K_{n,n,n}$.*

Stirling's approximation gives $n! = e^{n\ln n - n(1+o(1))}$. Thus $\frac{n!}{6n^2(n-1)} = e^{n\ln n - n(1+o(1))}$. We extend Theorem 1 as follows:

Theorem 2. *Let k be a constant, and let p be a prime number. Moreover, let $n = kp$. Then as $p \to \infty$ there are at least $e^{p\ln p - p(1+\ln k+o(1))}$ nonisomorphic embeddings $T(u,v;\alpha)$ of the complete tripartite graph $K_{n,n,n}$.*

A special case is the embedding $T(u,v;\alpha)$ with $\alpha = (0,1,\ldots,n-1)$, and with the vertices $W = \{w_0, w_1, \ldots, w_{n-1}\}$ placed into the faces of $B(u,v;\alpha)$ as described above. Then one colour class of triangles of $T(u,v;\alpha)$ contains triangles $u_j v_{j+k} w_k$, $0 \le j,k \le n-1$, while the other one contains triangles $v_j u_{j-k+1} w_k$. Thus, the corresponding Latin squares are

$$\begin{bmatrix} 0 & 1 & 2 & \ldots & n-1 \\ n-1 & 0 & 1 & \ldots & n-2 \\ n-2 & n-1 & 0 & \ldots & n-3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \ldots & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 3 & \ldots & 0 \\ 0 & 1 & 2 & \ldots & n-1 \\ n-1 & 0 & 1 & \ldots & n-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 3 & 4 & \ldots & 1 \end{bmatrix}$$

Observe that permuting the rows of both squares by the same permutation corresponds to relabelling the vertices $u_0, u_1, \ldots, u_{n-1}$. Applying the permutation $(0,1,2,3,\ldots,n-1) \to (1,0,n-1,n-2,\ldots,2)$ to rows of these squares we get the squares

$$\begin{bmatrix} n-1 & 0 & 1 & \ldots & n-2 \\ 0 & 1 & 2 & \ldots & n-1 \\ 1 & 2 & 3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n-2 & n-1 & 0 & \ldots & n-3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 2 & \ldots & n-1 \\ 1 & 2 & 3 & \ldots & 0 \\ 2 & 3 & 4 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n-1 & 0 & 1 & \ldots & n-2 \end{bmatrix}$$

Triples $u_i v_j w_k$ of the first table satisfy $k \equiv i+j-1 \pmod{n}$, while the triples of the other one satisfy $k \equiv i+j \pmod{n}$. In [4] it is proved that the biembedding of these squares is regular, so that $T(u,v;\alpha)$ with $\alpha = (0,1,\ldots,n-1)$ is a regular embedding as well. Recall here that an embedding M is regular if and only if for every two triples (z_1, e_1, f_1) and (z_2, e_2, f_2), where e_i is an edge incident to vertex z_i and face f_i, $1 \le i \le 2$, there exists an automorphism of M which maps z_1 to z_2, e_1 to e_2, and f_1 to f_2 (hence, we require also the automorphisms which reverse the global orientation of the surface). Based on these facts, we prove:

Theorem 3. *There is a unique regular triangular embedding of a complete tripartite graph $K_{n,n,n}$ in an orientable surface.*

Of course, by "unique" we mean "unique up to isomorphism".

3. Proofs

Consider the question: "Under which conditions are the embeddings $T(u,v;\alpha)$ and $T(u,v;\beta)$ isomorphic?" We address this question by examining possible isomorphisms of $B(u,v;\alpha)$ and $B(u,v;\beta)$. Obviously, if $B(u,v;\alpha)$ is isomorphic to $B(u,v;\beta)$, then $T(u,v;\alpha)$ is isomorphic to $T(u,v;\beta)$. Let $\alpha = (a_0, a_1, \ldots, a_{n-1})$.

Clearly, if $\beta = (a_{c+0}, a_{c+1}, \ldots, a_{c+(n-1)})$, i.e., if β is obtained by *rotating* α, then the corresponding voltage embeddings (of dipoles) in the sphere are identical. Hence, in this case the lifted embeddings are not only isomorphic, they are identical.

Now suppose that $\beta = (c{+}a_0, c{+}a_1, \ldots, c{+}a_{n-1})$, i.e., β is obtained by *adding* a constant c to every voltage of α, which we denote by $\beta = c + \alpha$. Then a mapping μ, such that $\mu(u_i) = u_i$ and $\mu(v_i) = v_{c+i}$, $0 \le i \le n{-}1$, is an isomorphism of $B(u,v;\alpha)$ onto $B(u,v;\beta)$.

Next, suppose that $\beta = (ca_0, ca_1, \ldots, ca_{n-1})$, i.e., β is obtained by *multiplying* the voltages of α by a constant c coprime with n, which we denote by $\beta = c\alpha$. Then a mapping μ, such that $\mu(u_i) = u_{ci}$ and $\mu(v_i) = v_{ci}$, $0 \le i \le n{-}1$, is an isomorphism of $B(u,v;\alpha)$ onto $B(u,v;\beta)$.

Finally, suppose that $\beta = (a_{n-1}, a_{n-2}, \ldots, a_0)$, i.e., β is obtained by *reversing* the permutation α, which we denote by $\beta = \alpha^{-1}$. Then $B(u,v;\beta)$ is isomorphic to $B(u,v;-\beta)$, where $-\beta = (-1)\beta = (-a_{n-1}, -a_{n-2}, \ldots, -a_0)$, so that it is enough to find an isomorphism μ mapping $B(u,v;\alpha)$ onto $B(u,v;-\beta)$. But this can be done by $\mu(u_i) = v_i$ and $\mu(v_i) = u_i$, $0 \le i \le n{-}1$.

We say that two permutations α and β are *equivalent*, if β can be obtained from α by a sequence of operations consisting of rotating, adding, multiplying and reversing. For permutations of n elements, denote by n_e the number of equivalence classes. It is shown below that there are at least $\lceil \frac{n_e}{3} \rceil$ nonisomorphic embeddings $T(u,v;\alpha)$ of $K_{n,n,n}$. The numbers n_e for small n, $7 \le n \le 16$, are presented in Table 1 (we remark that $n_e = 1$ for $n = 2, 3, 4, 6$, and $n_e = 2$ for $n = 5$). All these numbers were obtained by computer.

n	7	8	9	10	11	12	13	14	15	16
n_e	13	7	39	9	16,687	15	1,537,182	2,597	88,782	796,291

Table 1

As described above, if two permutations are equivalent, then the corresponding embeddings are isomorphic. In a sense we can reverse this statement:

Lemma 4. *If $B(u,v;\alpha)$ and $B(x,y;\beta)$ are isomorphic embeddings then α and β are equivalent permutations.*

Proof. First suppose that $B(u,v;\alpha)$ and $B(x,y;\beta)$ are isomorphic embeddings with isomorphism μ sending $\{U\}$ onto $\{X\}$ and $\{V\}$ onto $\{Y\}$.

Assume that $\mu(u_0) = x_{-k_r}$ for some $-k_r \in \mathbb{Z}_n$. In $B(x,y;\beta)$ consider a mapping ν such that $\nu(x_i) = x_{k_r+i}$ and $\nu(y_i) = y_{k_r+i}$, $0 \le i \le n{-}1$. Then ν is an automorphism of $B(x,y;\beta)$ which maps x_{-k_r} onto x_0. Hence, composing μ with ν we get an isomorphism μ_r mapping $B(u,v;\alpha)$ onto $B(x,y;\beta)$, such that $\mu_r(u_0) = x_0$.

Now assume that $\mu_r(v_0) = y_{-k_a}$. Then $B(x,y;\beta)$ is isomorphic to $B(x,y;k_a + \beta)$, and composing μ_r with this isomorphism we get an isomorphism μ_a mapping $B(u,v;\alpha)$ onto $B(x,y;k_a + \beta)$, such that $\mu_a(u_0) = x_0$ and $\mu_a(v_0) = y_0$.

Further, $\alpha = (\dots, -d, 0, \dots)$ for some d coprime with n. Assume that $\mu_a(u_d) = x_{d^*}$. Observe that the 2-gon of the voltage graph with vertices u and v and voltages 0 and $-d$ is lifted to a $2n$-gon with boundary cycle $u_0, v_0, u_d, v_d, \dots, u_{-d}, v_{-d}$ in $B(u, v; \alpha)$. As μ_a induces an isomorphism of $B(u, v; \alpha)$ onto $B(x, y; k_a + \beta)$, the image of this face is again a face and its boundary cycle is $x_0, y_0, x_{d^*}, y_{q_1}, x_{q_2}, \dots$. However, as all the faces of $B(x, y; k_a + \beta)$ are obtained by lifts of 2-gons of a voltage graph, we have $k_a + \beta = (\dots, -d^*, 0, \dots)$, where d^* is coprime with n. Thus, there is a multiplier k_m such that $k_m d^* = d$. Now $B(x, y; k_a + \beta)$ is isomorphic to $B(x, y; k_m(k_a + \beta))$, and composing μ_a with this isomorphism gives a new isomorphism μ_m mapping $B(u, v; \alpha)$ onto $B(x, y; \beta_m)$, $\beta_m = k_m(k_a + \beta)$, in which $\mu_m(u_0) = x_0$, $\mu_m(v_0) = y_0$, and $\mu_m(u_d) = x_d$. Moreover, the face $f : u_0, v_0, u_d, v_d, \dots, u_{-d}, v_{-d}$ is mapped onto the face $\mu_m(f) : x_0, y_0, x_d, y_d, \dots, x_{-d}$, y_{-d}, so that $\mu_m(u_i) = x_i$ and $\mu_m(v_i) = y_i$. Observe that all vertices of the embedded complete bipartite graphs appear on these two faces. It follows that $\alpha = \beta_m$, and, as β_m is equivalent to β, so is α.

Now suppose that $\mu(U) = Y$ and $\mu(V) = X$. As $B(x, y; \beta)$ is identical with $B(y, x; -\beta^{-1})$, the mapping μ takes $B(u, v; \alpha)$ onto $B(y, x; -\beta^{-1})$. Obviously, this reduces the case to the previous one. □

Observe that $T(u, v; \alpha)$ and $T(x, y; \beta)$ are isomorphic with an isomorphism sending $\{U, V\}$ onto $\{X, Y\}$ if and only if $B(u, v; \alpha)$ and $B(x, y; \beta)$ are isomorphic and therefore α and β are equivalent.

Proof of Theorem 1. Since n is a prime number, any permutation α of $\{0, 1, \dots, n-1\}$ gives an embedding $T(u, v; \alpha)$ of a complete tripartite graph. Equivalent permutations to α are obtained by taking either α or α^{-1}, applying one of n possible rotations, one of n possible additions of a constant, and one of $n-1$ possible multiplications by a non-zero constant. Hence the number of permutations equivalent to α is at most $2n^2(n-1)$. Thus, by Lemma 4, there are at most $2n^2(n-1)$ automorphisms of $T(u, v; \alpha)$ that send $\{U, V\}$ to $\{U, V\}$.

Suppose that k is the number of distinct automorphisms of $T(u, v; \alpha)$ that send $\{U, V\}$ to $\{U, W\}$, and denote these automorphisms by ϕ_i, $i = 1, 2, \dots, k$. Then, for each $i = 1, 2, \dots, k$, the mapping $\phi_i^{-1}\phi_1$ is an automorphism of $T(u, v; \alpha)$ that sends $\{U, V\}$ to $\{U, V\}$. Consequently $k \leq 2n^2(n-1)$.

Similarly, the number of distinct automorphisms of $T(u, v; \alpha)$ that send $\{U, V\}$ to $\{V, W\}$ is at most $2n^2(n-1)$. Thus the total number of automorphisms of $T(u, v; \alpha)$ is at most $6n^2(n-1)$. Hence, there are at least $\frac{n!}{6n^2(n-1)}$ nonisomorphic embeddings $T(u, v; \alpha)$. □

We remark that the lower bound $\lceil \frac{n!}{2n^2(n-1)} \rceil$ seems to be a good estimate of n_e, provided that n is a prime number. In Table 2 we compare these numbers for $n = 5, 7, 11$ and 13.

n	5	7	11	13
n_e	2	13	16,687	1,537,182
$\lceil \frac{n!}{2n^2(n-1)} \rceil$	1	9	16,495	1,535,262

Table 2

Proof of Theorem 2. In view of Theorem 1, we may assume that $k \geq 2$. Let $n = kp$ where $p > 3k+2$ and is a prime number. It is not easy to determine the number of permutations $\alpha = (a_0, a_1, \dots, a_{n-1})$, such that $a_i - a_{i-1}$ is always coprime with n. Therefore we consider only special permutations of this form.

Let $q = (k-1)p$. Further, let $\mathcal{A}$ be the set of all permutations $\alpha = (a_0, a_1, \dots, a_{n-1})$, such that $a_i = i$ whenever $i \leq q$; $a_j \equiv j \pmod{k}$ whenever $q+1 \leq j \leq n-1$; and $a_{q+1} = q+1+k$, $a_{n-1} = n-1-2k$. These choices for a_{q+1} and a_{n-1} are valid because $p > 3k+2$. Also note that $1 < k+1 = a_{q+1} - a_q < n - a_{n-1} = 2k+1$.

Consider any $\alpha \in \mathcal{A}$, $\alpha = (a_0, a_1, \dots, a_{n-1})$. For every i, $0 \leq i \leq n-1$, we have $a_i - a_{i-1} \equiv 1 \pmod{k}$. Here and elsewhere a_{-1} is interpreted as a_{n-1}. Moreover, $a_i - a_{i-1} \equiv 1 \pmod{p}$, $1 \leq i \leq q$; $|a_i - a_{i-1}| < p$, $q+2 \leq i \leq n-1$; $a_{q+1} - a_q < p$ and $a_0 - a_{n-1} \equiv 2k+1 \not\equiv 0 \pmod{p}$. Hence $a_i - a_{i-1} \not\equiv 0 \pmod{p}$, $0 \leq i \leq n-1$. Thus, $a_i - a_{i-1}$ is coprime with n for every i, $0 \leq i \leq n-1$, and hence, every $\alpha \in \mathcal{A}$ is a permutation such that $T(u, v; \alpha)$ is a triangular embedding of $K_{n,n,n}$.

Now we show that there are no equivalent permutations in $\mathcal{A}$. Let $\alpha \in \mathcal{A}$, $\alpha = (a_0, a_1, \dots, a_{n-1})$. Then $a_1 - a_0 = a_2 - a_1 = \cdots = a_q - a_{q-1} = 1$, and $q \geq n/2$. We also have $a_0 - a_{n-1} \equiv n - a_{n-1} \pmod{n}$ and $n - a_{n-1} > a_{q+1} - a_q > 1$, so that $a_0 - a_{n-1}$ and $a_{q+1} - a_q$ are distinct and neither are congruent to 1 modulo n.

Any finite sequence of the operations multiplication, reversal, rotation and addition applied to α may be reduced to a sequence of at most four operations, one of each type, in that order. To fix α by such a sequence, multiplication must be either by 1 with no subsequent reversal, or by -1 followed by a reversal; this is because any subsequent rotation and addition will not affect the differences $a_i - a_{i-1}$. In the former case the choices of $a_0, a_1, \dots, a_{n-1}$ ensure that the rotation and addition operations cannot fix α unless they reduce to the identity mapping. In the latter case, the rotation must be through $n-q-1$ places followed by addition of q in order to obtain a permutation $\beta = (b_0, b_1, \dots, b_{n-1})$ where $b_i = i$, $0 \leq i \leq q$. This gives $b_{q+1} = q+2k+1$ and $b_{n-1} = n-k-1$. Hence $\beta \notin \mathcal{A}$, so that $\mathcal{A}$ contains no permutations equivalent to α, apart from α itself. In fact, α is lexicographically the first permutation amongst all the permutations equivalent to α.

In each residue class modulo k there are either $\lfloor \frac{p-1}{k} \rfloor$ or $\lceil \frac{p-1}{k} \rceil$ numbers from $q+1$ to $n-1$. An immediate consequence is that $|\mathcal{A}| \leq (\lceil \frac{p-1}{k} \rceil !)^k$ and, after fixing a_{q+1} and a_{n-1}, we also have $|\mathcal{A}| \geq ((\lfloor \frac{p-1}{k} \rfloor - 2)!)^k$. By again applying Stirling's approximation, these give $\frac{|\mathcal{A}|}{3} = e^{p \log p - p(1 + \ln k + o(1))}$ as $p \to \infty$. □

Proof of Theorem 3. Let M be a regular face 2-colourable triangular embedding of a complete tripartite graph $K_{n,n,n}$. Then deleting independent vertices (and the edges incident with them) of one partite, we get from M a regular embedding M' of the complete bipartite graph $K_{n,n}$ in an orientable surface. The embedding M' has n faces, each of which contains all $2n$ vertices of the graph. Let f_0 be one of these faces. Denote the vertices of the boundary cycle of f_0 consecutively (say anti-clockwise) by

$$u_0, v_1, u_1, v_2, \dots, u_{n-1}, v_0.$$

In what follows we show that $M' = B(u, v; \alpha)$ for some permutation α.

Consider the clockwise rotation around u_0. In this rotation we have a sequence $\dots, v_0, v_1, v_{d+1}, \dots$ for some d. There is an automorphism ψ_k mapping (u_0, u_0v_1, f_0) onto (u_k, u_kv_{k+1}, f_0), $0 \leq k \leq n-1$. By considering the boundary of f_0, we find that $\psi_k(u_i) = u_{k+i}$ and $\psi_k(v_i) = v_{k+i}$. Hence in the clockwise rotation around

u_k we have $\ldots, v_k, v_{k+1}, v_{k+d+1}, \ldots$. There is also an automorphism χ_l mapping (u_0, u_0v_1, f_0) onto (v_l, v_lu_{l-1}, f_0), $0 \le l \le n{-}1$. By considering the boundary of f_0, we find that $\chi_l(u_i) = v_{l-i}$ and $\chi_l(v_i) = u_{l-i}$. Hence in the clockwise rotation around v_l we have $\ldots, u_{l-(d+1)}, u_{l-1}, u_l, \ldots$. It follows that there is a face f_1 with boundary cycle (reading anti-clockwise)

$$v_1, u_0, v_{d+1}, u_d, v_{2d+1}, u_{2d}, \ldots, u_{-d}.$$

Note that this implies that d is coprime with n.

There is an automorphism θ mapping (u_0, u_0v_1, f_0) onto (u_0, u_0v_{d+1}, f_1). By considering the boundaries of f_0 and f_1 we find that $\theta(u_i) = u_{di}$ and $\theta(v_i) = v_{di+1}$. Then by applying θ^j for $j = 1, 2, \ldots$, it follows that the clockwise rotation around u_0 has the form $\ldots, v_0, v_1, v_{d+1}, v_{d^2+d+1}, \ldots$. Applying ψ_k to this, we deduce that the clockwise rotation around u_k has the form $\ldots, v_k, v_{k+1}, v_{k+d+1}, v_{k+d^2+d+1}, \ldots$, and applying χ_l we see that the clockwise rotation around v_l has the form $\ldots, u_{l-(d^2+d+1)}, u_{l-(d+1)}, u_{l-1}, u_l, \ldots$. Note that these require the n numbers $0, 1, d{+}1, d^2{+}d{+}1, \ldots, d^{n-2}{+}d^{n-3}{+}\ldots{+}d{+}1$, to be distinct modulo n, and that $d^{n-1}{+}d^{n-2}{+}\ldots{+}d{+}1 \equiv 0 \pmod{n}$. It follows that $M' = B(u, v; \alpha)$ where $\alpha = (0, 1, d{+}1, \ldots, d^{n-2}{+}d^{n-3}{+}\ldots{+}d{+}1)$.

Now we show that $d = 1$. Denote the faces of M' around u_0 by $f_0, f_1, \ldots, f_{n-1}$ where $f_j = \theta^j(f_0)$, place into each face f_i a vertex w_i and join it to all the vertices lying on the boundary of f_i, to obtain, up to isomorphism, the original regular embedding $M = T(u, v; \alpha)$. Further, denote $D_k = d^k{+}d^{k-1}{+}\ldots{+}d{+}1$, $0 \le k \le n{-}1$. Then the rotations around u_k, v_k and w_k are as follows:

$$\begin{aligned}
u_k &: v_{k+D_{n-1}}{=}v_k, w_0, v_{k+D_0}, w_1, v_{k+D_1}, w_2, \ldots, v_{k+D_{n-2}}, w_{n-1}\\
v_k &: u_{k-D_{n-1}}{=}u_k, w_{n-1}, u_{k-D_{n-2}}, w_{n-2}, \ldots, w_2, u_{k-D_1}, w_1, u_{k-D_0}, w_0\\
w_k &: u_0, v_{-d^k+D_k}, u_{-d^k}, v_{-2d^k+D_k}, u_{-2d^k}, \ldots, u_{d^k}, v_{D_k}
\end{aligned}$$

Observe that the rotation around w_k is opposite to the boundary of f_k in M'.

The triples (u_0, v_1, w_0) and (w_0, u_{-1}, v_0) are triangles of the map M. As M is regular, there is an automorphism φ of M mapping $(u_0, u_0v_1, u_0v_1w_0)$ onto $(w_0, w_0u_{-1}, w_0u_{-1}v_0)$. We determine $\varphi(v_{d+1})$.

Recall that the second arc following u_0v_1 in the rotation around u_0 is u_0v_{d+1}. As the second arc following $w_0u_{-1} = w_0u_{-d^0}$ in the rotation around w_0 is $w_0u_{-2d^0} = w_0u_{-2}$, we have $\varphi(v_{d+1}) = u_{-2}$.

In the rotation around $v_{d+1} = v_{D_1}$ we have $\ldots, u_{d+1-D_1}{=}u_0, w_1, u_d, w_0, \ldots$ so that in the rotation around $\varphi(v_{d+1}) = u_{-2}$ we must have $\ldots, \varphi(u_0), \varphi(w_1), \varphi(u_d), \varphi(w_0), \ldots$. As $\varphi(u_0) = w_0$ and $\varphi(w_0) = v_0$, in the rotation around u_{-2} we have $\ldots, w_0, v_a, w_b, v_0, \ldots$ for some a and b. But in the rotation around u_{-2} we have $\ldots, w_0, v_{-2+D_0}, w_1, v_{-2+D_1}, \ldots$, so that $v_0 = v_{-2+D_1}$, and hence $0 = -2 + d + 1$. Thus, $d = 1$ and $M = T(u, v; \alpha)$, where $\alpha = (0, 1, \ldots, n{-}1)$. $\square$

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