A Steiner triple system which colours all cubic graphs

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Abstract

We prove that there is a Steiner triple system $\mathcal{T}$ such that every simple cubic graph can have its edges coloured by points of $\mathcal{T}$ in such a way that for each vertex the colours of the three incident edges form a triple in $\mathcal{T}$. This result complements the result of Holroyd and Škoviera that every bridgeless cubic graph admits a similar colouring by any Steiner triple system of order greater than 3. The Steiner triple system employed in our proof has order 381 and is probably not the smallest possible.

1 Introduction

It is well known that the edges of each cubic graph can be coloured by three or by four colours in such a way that adjacent edges receive distinct colours. An edge-colouring of a cubic graph with three colours, known as a *Tait colouring*, has the property that the colours of two adjacent edges determine the colour of the third edge incident with their common vertex. One way of viewing this is to regard the colours as the points of a Steiner triple system and the set of colours at each vertex as a triple of the Steiner triple system. In a Tait colouring all the vertices have the same triple of colours, but we may consider more general edge-colourings of cubic graphs using arbitrary Steiner triple systems where the edges at different vertices are allowed to be coloured by different triples. Such colourings are the main object of study in the present paper.

Recall that a *Steiner triple system* $\mathcal{S} = (X, B)$ of order n is a collection B of three-element subsets (called *triples*) of a set X of n *points* such that each pair of points is together present in exactly one triple. A necessary and sufficient condition for the existence of such a system is that $n \equiv 1$ or 3 (mod 6), and such values of n are called *admissible*.

Now we can formally define an *$\mathcal{S}$-colouring* of a cubic graph G, where $\mathcal{S}$ is a Steiner triple system, as a colouring of the edges of G by points of $\mathcal{S}$ such that the colours of any three edges meeting at a vertex form a triple of $\mathcal{S}$. If G admits such a colouring, then we say that it is *$\mathcal{S}$-colourable*.

The study of this sort of edge-colourings has been proposed by Archdeacon [A]. The general question can be stated as follows: *Which cubic graphs can be coloured by which Steiner triple systems?*

In [F], Fu identified two classes of cubic graphs which can be coloured by using the Steiner triple system of order 7, the point-line design of the Fano plane $PG(2, 2)$. The first of these consists of all bridgeless cubic graphs of order at most 189, and the second comprises all such graphs of genus at most 24. Recently, Holroyd and Škoviera [HS] obtained a substantial improvement of these results. In particular, they proved the following two theorems.

Theorem A *Let G be a bridgeless cubic graph without loops, and let $\mathcal{S}$ be a Steiner triple system of order greater than 3. Then G is $\mathcal{S}$-colourable.*

Theorem B *Let G be a cubic graph without loops and let $\mathcal{S}$ be a projective Steiner triple system, that is an n-dimensional projective geometry $PG(n, 2)$*

whose triples are the lines of the geometry. Then G is $\mathcal{S}$-colourable if and only if G is bridgeless.

Both theorems hold for bridgeless cubic graphs without loops but with possible multiple edges. However, it is equally natural to deal with Steiner colourings of graphs which may contain bridges. It is the aim of this paper to extend the study of Steiner colourings to cubic graphs with bridges.

When dealing with Steiner colourings of cubic graphs with bridges it is initially convenient to exclude parallel edges because a cubic graph containing a triangle with one edge doubled cannot be coloured by any Steiner triple system.

Theorem B shows that there are infinitely many Steiner triple systems which cannot colour any cubic graphs with bridges. This suggests the question of whether there exists at least one Steiner triple system $\mathcal{S}$ such that every simple cubic graph (i.e., one without loops and multiple edges) is colourable by $\mathcal{S}$. The purpose of this paper is to answer this question in the affirmative.

Main Theorem *There is a Steiner triple system $\mathcal{T}$ such that every simple cubic graph is $\mathcal{T}$-colourable.*

To prove this theorem we construct a Steiner triple system $\mathcal{T}$ on 381 points and show that it can be used to colour every simple cubic graph. We leave it as an open question whether there exist smaller systems with a similar property.

As a corollary to the Main Theorem, we characterize those non-simple cubic graphs which admit no $\mathcal{S}$-colouring for any Steiner triple system $\mathcal{S}$, and we prove that those which do not admit an $\mathcal{S}$-colouring for some $\mathcal{S}$ are, in fact, $\mathcal{T}$-colourable.

2 Construction of the Steiner triple system

In this section we construct a Steiner triple system $\mathcal{T}$ on 381 points such that every cubic graph is $\mathcal{T}$-colourable. We use two general constructions to obtain the system $\mathcal{T}$.

The first construction is known as a *Pasch switch*. Let $\mathcal{S} = (X, B)$ be a Steiner triple system. A *Pasch configuration* in $\mathcal{S}$ is a configuration of four triples from B that cover precisely six points. Up to isomorphism, there is a unique configuration of this type and it has the form $\{\{a_0, b_0, c_0\}, \{a_0, b_1, c_1\}, \{a_1, b_0, c_1\}, \{a_1, b_1, c_0\}\}$. Given such a configuration, the four triples may be replaced by the triples $\{a_1, b_1, c_1\}$, $\{a_1, b_0, c_0\}$, $\{a_0, b_1, c_0\}$ and $\{a_0, b_0, c_1\}$ to produce a new Steiner triple system. The replacement operation is called a *Pasch switch*.

The second construction is the *tripling construction*. Given a Steiner triple system $\mathcal{S}$ on n points, the tripling construction produces a new Steiner triple system $\mathcal{S}^3$ on $3n$ points. The system $\mathcal{S}^3$ consists of three copies of $\mathcal{S}$, say $\mathcal{S}$, $\mathcal{S}'$ and $\mathcal{S}''$, its triples being those of each individual copy of $\mathcal{S}$ together with

the six triples of the form $\{a, b', c''\}$ for each triple $\{a, b, c\}$ of $\mathcal{S}$,

all triples of the form $\{a, a', a''\}$ for each point a of $\mathcal{S}$,

where $a', b', c', \ldots$ denote points of $\mathcal{S}'$, and $a'', b'', c'', \ldots$ denote points of $\mathcal{S}''$.

Now we can describe the construction of $\mathcal{T}$. We start with the 6-dimensional projective geometry $\mathcal{P} = PG(6, 2)$ over the field of two elements, which we view as a Steiner triple system of order $2^7 - 1$ whose triples are the lines of $\mathcal{P}$. Thus the points of $\mathcal{P}$ are the non-zero binary vectors of length 7 with entries from $\mathbb{Z}_2$, and the triples of the system are those triples of vectors whose sum is the zero vector.

Let us take the following six points of $\mathcal{P}$:

$$\begin{aligned}
&\boldsymbol{a}_0 = (1,1,0,1,1,0,0), \quad \boldsymbol{b}_0 = (1,0,0,1,0,1,0), \quad \boldsymbol{c}_0 = (0,1,0,0,1,1,0),\\
&\boldsymbol{a}_1 = (1,1,0,1,1,0,1), \quad \boldsymbol{b}_1 = (1,0,0,1,0,1,1), \quad \boldsymbol{c}_1 = (0,1,0,0,1,1,1).
\end{aligned}$$

The triples $\{\boldsymbol{a}_0, \boldsymbol{b}_0, \boldsymbol{c}_0\}$, $\{\boldsymbol{a}_0, \boldsymbol{b}_1, \boldsymbol{c}_1\}$, $\{\boldsymbol{a}_1, \boldsymbol{b}_0, \boldsymbol{c}_1\}$ and $\{\boldsymbol{a}_1, \boldsymbol{b}_1, \boldsymbol{c}_0\}$ form a Pasch configuration in $\mathcal{P}$. So we can perform the Pasch switch on these four triples to produce a new system $\mathcal{Q}$. The final Steiner triple system $\mathcal{T}$ is formed by tripling $\mathcal{Q}$, that is, $\mathcal{T} = \mathcal{Q}^3$; its order is $3(2^7 - 1) = 381$.

Our next aim is to show that every simple cubic graph can be coloured by $\mathcal{T}$. The colourings which we will construct only use a part of the system $\mathcal{T}$. Let us choose the point $\boldsymbol{z} = (0,0,0,0,0,0,1) \in \mathcal{Q}$, and let $\boldsymbol{z}'$ and $\boldsymbol{z}''$ be its copies in $\mathcal{Q}'$ and $\mathcal{Q}''$, respectively. Denote by $\mathcal{Z}$ the trivial Steiner triple system consisting of a single triple $\{\boldsymbol{z}, \boldsymbol{z}', \boldsymbol{z}''\}$. Note that $\mathcal{Z} \subseteq \mathcal{T}$. Besides this triple, we will use only triples within $\mathcal{Q}$ and its copies $\mathcal{Q}'$ and $\mathcal{Q}''$.

3 Proof

We start with a brief outline of the proof. Given a cubic graph we decompose it into 2-connected blocks and bridges. We distinguish between two types of 2-connected blocks – those containing a subdivision of the complete graph K_4 on four vertices, and those not. The latter ones are graphs known to belong to the family of so-called *series-parallel* graphs. As we shall see later, each series-parallel graph with maximum valency 3 is 3-edge-colourable. Thus it can be coloured by $\mathcal{Z}$. Since $\mathcal{Z} \subseteq \mathcal{T}$, this part of the colouring has the required property.

Clearly, individual colourings of each series-parallel block will force the colours of some bridges to be within $\mathcal{Z}$. This leads us to the decision to colour *each* bridge by one of the colours $\boldsymbol{z}$, $\boldsymbol{z}'$ and $\boldsymbol{z}''$.

Now we consider a 2-connected block which contains a subdivision of K_4. We suppress its 2-valent vertices to obtain a cubic graph which, by Theorem A, is colourable by the Fano plane $PG(2,2)$. We therefore choose in $\mathcal{Q}$ the set of all points $(x,y,z,0,0,0,0)$ where $x,y,z \in \mathbb{Z}_2$ and $(x,y,z) \neq (0,0,0)$, and form the triples of vectors whose sum is the zero vector. The resulting configuration $\mathcal{F}$ in $\mathcal{Q}$ is isomorphic to the Fano plane. We colour the cubic graph in question by $\mathcal{F}$ and "subdivide the colouring" into an improper colouring (i.e., a colouring where some adjacent edges have the same colour) of the original block. Then we colour the incident bridges by $\boldsymbol{z}$ and modify the latter colouring into a proper $\mathcal{Q}$-colouring. This modification will make use of the subdivision of K_4 and will influence the other four coordinates of the vectors. In fact, the first three of them will depend on a selected 3-valent vertex of the subdivision of K_4, while the last coordinate localizes the special point $\boldsymbol{z}$ of $\mathcal{Q}$.

The next part of this section provides several results preparing the main proof, especially the crucial Lemmas 1 and 2. Throughout the rest of this section graphs will always be loopless but may have parallel edges unless we specify that they are simple.

Recall that a graph is series-parallel if it can be constructed from the complete graph K_2 by a repeated use of two operations (which have given name to this family): (1) subdividing an edge, and (2) adding an edge parallel to an existing edge (i.e., increasing the multiplicity of an existing edge).

It is now part of folklore in graph theory that series-parallel graphs coincide with those having no subdivision of the complete graph K_4. The latter property can conveniently be taken as a definition of series-parallel graphs and the former definition as their characterization.

In order to be precise, let us call a graph *series-parallel* if it contains no subdivision of K_4. In [D], the following characterization of 2-connected series-parallel graphs has been established.

Theorem C *A* 2*-connected graph* G *is series-parallel if and only if it can be reduced to a loop by a sequence of the following two operations:*

S. *Replacing a path of length two with interior vertex of valency* 2 *by a single edge.*

P. *Deleting an edge parallel to another edge.*

Unfortunately, the operation **S** can create multiple (i.e., parallel) edges even when the original graph G was simple. For graphs with maximum valency 3 this inconvenience can be avoided as follows.

Let G be a simple 2-connected series-parallel graph with maximum valency 3. Clearly, the length of each cycle in G is at least 3. If G contained only one vertex of valency 3, then it could not be 2-connected. Also, if every cycle of G contained three or more vertices of valency 3, then it would be impossible to reduce G to a loop using only operations **S** and **P**, because **P** could not be applied. Hence G contains a cycle with exactly two vertices of valency 3. Let C be a shortest such cycle. Then the edges which connect $G - C$ to C are distinct, and contracting C into a single vertex results in a 2-connected series-parallel graph which is either simple or is a cycle of length 2, that is, a pair of parallel edges. As the resulting graph is smaller than G, we have derived the following corollary of Theorem C:

Corollary 1 *A simple* 2*-connected series-parallel graph* G *with maximum valency at most* 3 *can be reduced to a cycle of length at least* 2 *by a repeated use of the following operation:*

SP. *Contracting a shortest cycle with exactly two vertices of valency* 3 *into a vertex.*

Moreover, at each instance, the length of the shortest cycle subject to contraction is at least 3.

We proceed to the edge-colouring lemmas.

Lemma 1 *A simple* 2*-connected series-parallel graph with maximum valency at most* 3 *is* 3*-edge-colourable.*

Proof. We employ induction on the number of vertices. Let G be a simple 2-connected series-parallel graph with maximum valency at most 3. If G is a cycle, then it is obviously 3-edge colourable. Otherwise G contains a cycle C with exactly two vertices of valency 3 such that the contracted graph G/C is either a simple 2-connected series-parallel graph with valency at most 3, or a cycle of length 2. In both these cases let u and v be the two 3-valent vertices on C and let x and y be the respective edges at u and v which are not on C. By the induction hypothesis, G/C admits a 3-edge-colouring ϕ in which x and y necessarily receive distinct colours, say $\phi(x) = 1$ and $\phi(y) = 2$. We now extend the colouring ϕ to a 3-edge-colouring of the whole G. The vertices u and v divide C into two independent u-v-paths $P = e_1e_2\ldots e_r$ and $Q = f_1f_2\ldots f_s$ with $r \leq s$. If $r = s = 2$, then we can set $\phi(e_1) = 2$, $\phi(e_2) = 3$, $\phi(f_1) = 3$ and $\phi(f_2) = 1$. Otherwise, if r is odd (possibly $r = 1$) we set $\phi(e_1) = \phi(e_r) = 3$, $\phi(f_1) = 2$, and $\phi(f_s) = 1$; if r is even (and $s \geq 3$), then we set $\phi(e_1) = 2$, $\phi(e_r) = 1$, and $\phi(f_1) = \phi(f_s) = 3$. The rest of the paths, and hence the whole of G, is easily coloured with three colours. □

Remark. Lemma 1 is false for graphs with parallel edges allowed. For example, a triangle with one edge doubled is a series-parallel graph with maximum valency 3, but has no 3-edge-colouring.

We call a graph *almost cubic* if all its vertices have valency 3 or 1. For an almost cubic graph G, let $G^\natural$ denote the graph obtained from G by deleting the vertices of valency 1 and their incident edges.

A *colouring* of an almost cubic graph by a Steiner triple system $\mathcal{S}$ is any colouring by points of $\mathcal{S}$ such that the colours of edges incident with a common 3-valent vertex form a triple of $\mathcal{S}$.

Lemma 2 *Let* G *be a connected almost cubic graph such that* $G^\natural$ *is* 2*-connected and contains a subdivision of* K_4*. Then* G *is colourable by* $\mathcal{Q}$ *in such a way that all the bridges of* G *receive the colour* $\boldsymbol{z}$*.*

Proof. Let H be the graph obtained from $G^\natural$ by suppressing all 2-valent vertices. Then H is a 2-connected cubic graph, not necessarily simple.

By Theorem A, H is colourable by the Fano plane. So we can colour H by $\mathcal{F} \subseteq \mathcal{Q}$. If $H = G$, this yields the required $\mathcal{Q}$-colouring of G.

Assume that $H \neq G$. Then $G - G^\natural$ consists of 1-valent vertices $u_1, u_2, \ldots, u_n$ which are connected to $G^\natural$ by pendant edges $u_1v_1, u_2v_2, \ldots, u_nv_n$. Take any $\mathcal{F}$-colouring ϕ_0 of H. Since $G^\natural$ arises from H by inserting the vertices $v_1, \ldots, v_n$ into certain edges, the subdivision operation transforms ϕ_0 into an improper colouring ϕ_1 of $G^\natural$ such that each edge arising by a subdivision of an edge of H inherits the colour from it. We will further modify this colouring into a $\mathcal{Q}$-colouring of G.

Since $G^\natural$ contains a subdivision of K_4, there is a vertex x and 3-cycles P, Q and R in $G^\natural$ such that $P \cap Q \cap R = \{x\}$. (Observe that any three distinct 3-cycles in K_4 intersect in exactly one vertex.)

Let us arrange the vertices $v_1, \ldots, v_n$ into $d = \lfloor n/2 \rfloor$ pairs; if n is odd we leave v_n unpaired. For $i = 1, 2, \ldots, d$ we pick any path L_i joining the i-th pair, and for n odd we let L_0 be a path joining v_n to x.

Now we construct a colouring ϕ_2 of G as follows. Let $c_i : E(G^\natural) \to \mathbb{Z}_2$ be the indicator function such that $c_i(e) = 1$ if and only if e lies on the path L_i. Note that if n is even, c_0 is identically zero. We colour each pendant edge $u_1v_1, \ldots, u_nv_n$ by $\boldsymbol{z} = (0,0,0,0,0,0,1)$, and for $e \in E(G^\natural)$ we set

$$\phi_2(e) = \phi_1(e) + \left(\sum_{i=0}^{d} c_i(e) \right) \boldsymbol{z}.$$

In other words, for each i we increase the colours of the edges on L_i by $\boldsymbol{z}$. If v_j is an end-vertex of L_i, then adding $\boldsymbol{z}$ to the colours of edges of L_i causes the edges of $G^\natural$ incident with v_j to receive different colours, and as $\phi_2(v_ju_j) = \boldsymbol{z}$, the edges incident with v_j in G are coloured by a triple of $\mathcal{Q}$. Note that if $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ is a triple of $\mathcal{P}$, and $\boldsymbol{d} \in \mathcal{P}$ is different from both $\boldsymbol{a}$ and $\boldsymbol{b}$, then $\{\boldsymbol{a}+\boldsymbol{d}, \boldsymbol{b}+\boldsymbol{d}, \boldsymbol{c}\}$ is again a triple of $\mathcal{P}$. Hence, if n is even, ϕ_2 is a proper $\mathcal{Q}$-colouring of G, while if n is odd, x is the unique vertex of G at which the colours of incident edges do not form a triple of $\mathcal{Q}$.

So assume that n is odd. Let xp, xq and xr be the edges incident with the vertex x such that the vertex p lies on $Q \cap R$, q lies on $P \cap R$, and r lies on $P \cap Q$. We now construct a colouring ϕ_3 of G as follows. Let $c_P : E(G) \to \mathbb{Z}_2$ be the indicator function such that $c_P(e) = 1$ precisely when e lies on the cycle P; for the other two cycles Q and R define c_Q and c_R similarly. The construction of ϕ_2 implies that its values on the edges incident with x have the form

$$\phi_2(xp) \;=\; (p_1, p_2, p_3, 0, 0, 0, p_4),$$

$$
\begin{aligned}
\phi_2(xq) &= (q_1, q_2, q_3, 0, 0, 0, q_4),\\
\phi_2(xr) &= (r_1, r_2, r_3, 0, 0, 0, r_4),
\end{aligned}
$$

where $p_i + q_i + r_i = \begin{cases} 0 & \text{if } \ i = 1, 2, 3,\\ 1 & \text{if } \ i = 4.\end{cases}$

Set

$$
\begin{aligned}
\boldsymbol{p} &= (q_1 + 1, q_2, q_3, 0, 0, 1, 0),\\
\boldsymbol{q} &= (p_1 + 1, p_2 + 1, p_3, 0, 1, 0, 0),\\
\boldsymbol{r} &= (0, 0, 0, 1, 0, 0, 0),
\end{aligned}
$$

and for each edge e of G we define

$$\phi_3(e) = \phi_2(e) + c_P(e)\boldsymbol{p} + c_Q(e)\boldsymbol{q} + c_R(e)\boldsymbol{r}.$$

Clearly, $\phi_3(xp) = (1, 1, 0, 1, 1, 0, p_4)$, $\phi_3(xq) = (1, 0, 0, 1, 0, 1, q_4)$, and $\phi_3(xr) = (0, 1, 0, 0, 1, 1, r_4)$. Observe that at each 3-valent vertex of $G - x$, the colours of the three incident edges all have their fourth, or fifth, or sixth entry zero. Thus at each 3-valent vertex different from x the colours of ϕ_3 form a triple of $\mathcal{Q}$. By the definition of ϕ_2, $p_4 + q_4 + r_4 = 1$, so the edges incident with x are also coloured by a triple of $\mathcal{Q}$, in fact one of the four triples of the Pasch configuration which was created in forming $\mathcal{Q}$ from $\mathcal{P}$. Hence, ϕ_3 is a proper $\mathcal{Q}$-colouring of G, as required. □

Now we are ready to prove the main result.

Proof of Main Theorem. Let G be a simple cubic graph, and let $B_1, \ldots, B_n$ be the components obtained from G by removing all bridges. We can assume that the components B_i are labelled in such a way that for each $i \geq 2$, there is exactly one bridge connecting B_i to $B_1 \cup \ldots \cup B_{i-1}$.

Let H_i be the graph obtained from B_i by adding all bridges of G which are incident with B_i. Clearly, H_i is an almost cubic graph. We colour the graphs $H_1, \ldots, H_n$ step by step with the increasing index. If $H_i^\natural = B_i$ is a series-parallel graph or a single vertex, then we 3-colour H_i by $\mathcal{Z}$. This is possible by Lemma 1. If B_i contains a subdivision K_4, then we employ Lemma 2 to colour H_i by a copy of $\mathcal{Q}$.

Assume that the graph $H_1 \cup \ldots \cup H_{k-1}$, $k \geq 2$, has already been coloured as indicated above. By the ordering of the graphs B_i, exactly one edge of H_k

– a bridge – has received its colour $\boldsymbol{u} \in \mathcal{Z}$ in one of the previous steps. If H_k contains a subdivision of K_4, then we colour H_k by that copy of $\mathcal{Q} \subseteq \mathcal{T}$ which contains $\boldsymbol{u}$. If H_k contains no subdivision of K_4, then we can easily extend the colouring of the bridge to an $\mathcal{Z}$-colouring of H_k. In both cases, $H_1 \cup \ldots \cup H_k$ is properly $\mathcal{T}$-coloured in such a way that the bridges receive a colour in $\mathcal{Z}$. Therefore we can continue the process until the whole of G is $\mathcal{T}$-coloured. □

Certain loopless cubic graphs which have parallel edges may also be coloured by $\mathcal{T}$. Indeed, it is possible to characterize these precisely by the following Corollary.

Corollary 2 *Every loopless cubic graph that does not have, as a subgraph, a cubic series-parallel graph with a single subdivided edge, is $\mathcal{T}$-colourable. Furthermore, any cubic graph that does have, as a subgraph, a cubic series-parallel graph with a single subdivided edge, is not $\mathcal{S}$-colourable for any Steiner triple system $\mathcal{S}$.*

Proof. Take a loopless cubic graph G. If G is simple then, by the Main Theorem, G is $\mathcal{T}$-colourable. Otherwise, if G has a triple edge xy, then this edge and the vertices x, y lie in a disconnected component H and we put $G' = G \setminus H$. If G has a double edge xy, then define the graph G' by deleting from G this double edge and the vertices x, y, and replacing the remaining two edges incident with x and y, say ux and vy, by a single edge uv. The possibility that $u = v$ is not excluded. Clearly, for any Steiner triple system $\mathcal{S}$, G is $\mathcal{S}$-colourable if and only if G' is $\mathcal{S}$-colourable. By repeatedly removing triple and double edges in this fashion, we arrive at a graph G^* without parallel edges such that G is $\mathcal{S}$-colourable if and only if G^* is $\mathcal{S}$-colourable.

The first part of the Corollary then follows from the Main Theorem, provided that G^* is loopless. But, if G^* contains a loop on a vertex v then, by reversing the above operations, it is easy to see that G must contain, as a subgraph, a cubic series-parallel graph with a single subdivided edge; indeed the unique subdivision point is v and the additional edge incident with v forms a bridge in G. To establish the second part of the Corollary, observe that a subgraph of G which comprises a cubic series-parallel graph with a single subdivided edge, reduces to a loop in G^*. □

4 Concluding remarks

Let G be a cubic graph. Define the *Steiner chromatic number* of G to be the smallest n for which there exists a Steiner triple system $\mathcal{S}$ of order n such that G admits an $\mathcal{S}$-colouring. With this definition, our main result, combined with Corollary 2, shows that every loopless cubic graph that does not have, as a subgraph, a cubic series-parallel graph with a single subdivided edge, has a finite Steiner chromatic number. Furthermore, Theorem A implies that a bridgeless cubic graph has Steiner chromatic number 3 or 7 according to whether or not it is 3-edge-colourable.

For graphs with bridges we have not been able to give such a simple description of their Steiner chromatic number. Since the only thing which we now know is just a rough general upper bound 381, we believe that the Steiner chromatic number is worth further investigation.

Our final observation uses the result of Doyen and Wilson [DW] that a Steiner triple system of order n may be embedded in a Steiner triple system of order m for any admissible $m \geq 2n + 1$. Consequently, for any admissible $m \geq 763$, there is a Steiner triple system of order m which will colour every simple cubic graph.

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