CONNECTIVITY OF PATH GRAPHS.

Martin Knor[1)] and Ľudovít Niepel[2)]

[1)] Slovak University of Technology, Faculty of Civil Engineering,
Department of Mathematics, Radlinského 11, 813 68 Bratislava, Slovakia,
E-mail: knor@vox.svf.stuba.sk;

[2)] Kuwait University, Faculty of Science,
Department of Mathematics & Computer
Science, P.O. box 5969 Safat 13060, Kuwait,
E-mail: NIEPEL@MATH-1.sci.kuniv.edu.kw.

Abstract. We prove a necessary and sufficient condition under which a connected graph has a connected P_3-path graph. Moreover, an analogous condition for connectivity of the P_k-path graph of a connected graph which does not contain a cycle of length smaller than $k+1$ is derived.

1. Introduction

Let G be a graph, $k \geq 1$, and let $\mathcal{P}_k$ be the set of all paths of length k (i.e., with $k+1$ vertices) in G. The vertex set of a **path graph** $P_k(G)$ is the set $\mathcal{P}_k$. Two vertices of $P_k(G)$ are joined by an edge if and only if the edges in the intersection of the corresponding paths form a path of length $k-1$, and their union forms either a cycle or a path of length $k+1$. It means that the vertices are adjacent if and only if one can be obtained from the other by "shifting" the corresponding paths in G.

Path graphs were investigated by Broersma and Hoede in [2] as a natural generalization of line graphs, since $P_1(G)$ is the line graph of G. We have to point out that, in the pioneering paper [2] the number k in $P_k(G)$ denotes the number of vertices of the paths and not their length. However, in some applications our notation is more consistent, see e.g. [3]. Traversability of P_2-path graphs is studied in [9], and a characterization of P_2-path graphs is given in [2] and [7]. Distance properties of path graphs are studied in [1], [4] and [5], and [6] and [8] are devoted to isomorphisms of path graphs.

1991 *Mathematics Subject Classification.* 05C40, 05C38.

Key words and phrases. Connectivity, path graph, cycle.

[1)] Supported by VEGA grant 1/6293/99

[2)] Supported by Kuwait University grant #SM 172

Let $V = V(G)$ be a set of n distinct symbols. Consider strings of length $k+1$ of these symbols, in which all $k+1$ symbols are mutually distinct. Let G be a graph on vertex set V, edges of which correspond to pairs of symbols which can be neighbours in our strings. If we do not distinguish between a string and its reverse, then $P_k(G)$ is connected if and only if every string can be obtained from any other one sequentially, by removing a symbol from one of its ends and adding a symbol to the other end.

Let G be a connected graph. It is well-known (and trivial to prove) that $P_1(G)$, i.e., the line graph of G, is a connected graph. However, this is not the case for P_k-path graphs if $k \geq 2$. This causes some problems, especially when studying distances in path graphs. For example, in [1] the authors give an upper bound for the diameter of every component of a P_k-path graph, as the whole graph can be disconnected. By [4, Theorem 1], we have:

Theorem A. *Let G be a connected graph. Then $P_2(G)$ is disconnected if and only if G contains two distinct paths A and B of length two, such that the degrees of both endvertices of A are 1 in G.*

In this paper we generalize Theorem A to P_k-path graphs when G does not contain a cycle of length smaller than $k+1$. Moreover, we completely solve the case of P_3-path graphs.

We use standard graph-theoretic notation. Let G be a graph. The vertex set and the edge set of G, respectively, are denoted by $V(G)$ and $E(G)$. For two subgraphs, H_1 and H_2 of G, by $H_1 \cup H_2$ we denote the union of H_1 and H_2, and $H_1 \cap H_2$ denotes their intersection. Let u and v be vertices in G. By $d_G(u, v)$ we denote the distance from u to v in G, and by $deg_G(u)$ the degree of u is denoted. For the vertex set of a component of G containing u we use $Co(u)$. A path and a cycle, respectively, of length l are denoted by P_l and C_l.

The outline of the paper is as follows. In Section 2 we give a (necessary and sufficient) condition for a connected graph (under some restrictions) to have a connected P_k-path graph, and Section 3 is devoted to an analogous condition for P_3-path graphs of general graphs.

2. P_k-PATH GRAPHS

Let G be a graph, $k \geq 2$, $0 \leq t \leq k-2$, and let A be a path of length k in G. By $P^*_{k,t}$ we denote an induced subgraph of G which is a tree of diameter $k+t$ with a diametric path $(x_t, x_{t-1}, \ldots, x_1, v_0, v_1, \ldots, v_{k-t}, y_1, y_2, \ldots, y_t)$, such that all endvertices of $P^*_{k,t}$ have distance $\leq t$ either to v_0 or to v_{k-t} and the degrees of $v_1, v_2, \ldots, v_{k-t-1}$ are 2 in $P^*_{k,t}$. Moreover, no vertex of $V(P^*_{k,t}) - \{v_1, v_2, \ldots, v_{k-t-1}\}$ is joined by an edge to a vertex in $V(G) - V(P^*_{k,t})$. The path $(v_0, v_1, \ldots, v_{k-t})$ is a **base** of $P^*_{k,t}$, and we say that A lies in $P^*_{k,t}$, $A \in P^*_{k,t}$, if and only if the base of $P^*_{k,t}$ is a subpath of A.

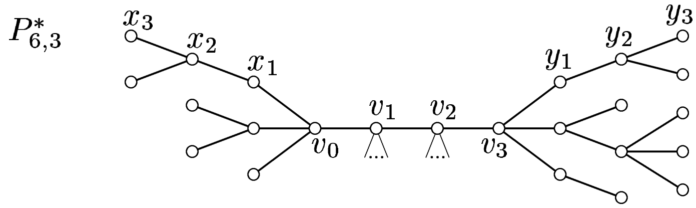

Figure 1

In Figure 1 a $P^*_{6,3}$ is pictured. Note that this graph contains also two $P^*_{6,0}$ and one $P^*_{6,1}$, but it does not contain a $P^*_{6,2}$. We remark that by thin halfedges are outlined possible edges joining vertices of $P^*_{6,3}$ to vertices in $V(G) - V(P^*_{6,3})$.

In this section we prove the following theorem.

Theorem 1. *Let G be a connected graph without cycles of length smaller than $k+1$. Then $P_k(G)$ is disconnected if and only if G contains a $P^*_{k,t}$, $0 \le t \le k-2$, and a path A of length k such that $A \notin P^*_{k,t}$.*

For easier handling of paths of length k in G (i.e., the vertices of $P_k(G)$) we adopt the following convention. We denote the vertices of $P_k(G)$ (as well as the vertices of G) by small letters a, b, ... , while the corresponding paths of length k in G will be denoted by capital letters A, B, ... It means that if A is a path of length k in G and a is a vertex in $P_k(G)$, then a must be the vertex corresponding to the path A.

Lemma 2. *Let G be a connected graph without cycles of length smaller than $k+1$. Moreover, let $A = (x_0, x_1, \dots, x_k)$ be a path of length k in G which is not in a $P^*_{k,t}$, $0 \le t \le k-2$. Then for every i, $0 \le i \le k$, there is an $a_i \in Co(a)$ such that x_i is an endvertex of A_i and the edge of A_i incident with x_i lies in A.*

Proof. Observe that if there is an $a_i \in Co(a)$ such that x_i is an endvertex of A_i, then choosing a_i with $d_{P_k(G)}(a, a_i)$ smallest possible, the endedge of A_i incident with x_i is in A.

Thus, suppose that for some i, $0 < i < k$, there is no $a_i \in Co(a)$ such that x_i is an endvertex of A_i. Let H be a subgraph of G formed by the vertices and edges of paths A', where $a' \in Co(a)$. Clearly, $(x_{i-1}, x_i, x_{i+1}) \subseteq A'$ for every $a' \in Co(a)$. Let $R = (v_0, v_1, \dots, v_{k-t})$ be the longest path that share all A', $a' \in Co(a)$. As $k-t \ge 2$, we have $t \le k-2$. Further, $deg_H(v_1) = deg_H(v_2) = \dots = deg_H(v_{k-t-1}) = 2$, and every endvertex of H has distance $\le t$ either to v_0 or to v_{k-t}. Since H does not contain cycles (recall that the length of every cycle in G is at least $k+1$), H is a $P^*_{k,t}$, $0 \le t \le k-2$. As $R \subseteq A$ we have $A \in P^*_{k,t}$, a contradiction. □

Let A and B be two paths of length k in G. If one endvertex of B, say x, lies in A, but the edge of B incident with x is not in A, then we say that the pair (A, B) forms a T with a touching vertex x.

Note that if (A, B) forms a T in G, then $A \cup B$ is not necessarily a tree even if G does not contain a cycle of length $\le k$.

Lemma 3. *Let G be a graph without cycles of length smaller than $k+1$. Moreover, suppose G does not contain a $P^*_{k,t}$, $0 \le t \le k-2$, and let (A, B) form a T in G. Then $b \in Co(a)$.*

Proof. Let (A, B) form a T with a touching vertex x. By Lemma 2, there is an $a' \in Co(a)$ such that x is an endvertex of A' and the edge of A' incident with

x lies in A. As G does not contain a cycle of length smaller than $k+1$, we have $d_{P_k(G)}(a', b) \le k$, and hence $b \in Co(a)$. $\square$

Now we are able to prove Theorem 1.

Proof of Theorem 1. We arrange the proof into three steps.

(i) First suppose that G contains a $P^*_{k,t}$, $0 \le t \le k-2$, with a base $R = (v_0, v_1, \ldots, v_{k-t})$, and a path A of length k such that $A \notin P^*_{k,t}$. Since the diameter of $P^*_{k,t}$ is $k+t$, there is a path B of length k in G such that $B \in P^*_{k,t}$, i.e., $R \subseteq B$. By the structure of $P^*_{k,t}$, for every vertex b' of $P_k(G)$ which is adjacent to b we have $R \subseteq B'$, too. Hence, for every $b' \in Co(b)$ it holds $R \subseteq B'$. Since A does not contain R, we have $a \notin Co(b)$, so that $P_k(G)$ is a disconnected graph.

(ii) Now suppose that G contains a $P^*_{k,t}$, $0 \le t \le k-2$, such that for every $a \in V(P_k(G))$ it holds $A \in P^*_{k,t}$. We show that either $P_k(G)$ is a connected graph, or G contains a $P^*_{k,t'}$, $0 \le t' < t$, and a path B of length k such that $B \notin P^*_{k,t'}$.

Let $R = (v_0, v_1, \ldots, v_{k-t})$ be the base of $P^*_{k,t}$, and let b be a vertex of $P_k(G)$ such that $B \in P^*_{k,t}$ and v_0 is an endvertex of B (e.g., choose B as a part of a diametric path of $P^*_{k,t}$). Let a be a vertex of $P_k(G)$, $A \in P^*_{k,t}$. If there is an $a' \in Co(a)$ such that either v_0 or v_{k-t} is an endvertex of A', then either $d_{P_k(G)}(a', b) \le 2t$ or $d_{P_k(G)}(a', b) = t$ (by the structure of $P^*_{k,t}$ we have $R \subseteq A'$). Hence, $a \in Co(b)$.

Thus, suppose that there is a vertex a in $P_k(G)$, $A \in P^*_{k,t}$, such that for every $a' \in Co(a)$ neither v_0 nor v_{k-t} is an endvertex of A'. Let H be a subgraph of G formed by the vertices and edges of paths A', for which $a' \in Co(a)$. Clearly, $R \subseteq A'$ for every $a' \in Co(a)$. Let $R' = (v'_0, v'_1, \ldots, v'_{k-t'})$ be the longest path that share all A', $a' \in Co(a)$. Since $R \subset R'$, by the choice of A we have $v_0 = v'_i$, $v_1 = v'_{i+1}$, $\ldots$, $v_{k-t} = v'_{i+k-t}$, where $i > 0$ and $i + k - t < k - t'$, i.e. $t' < t - i$. Further, $deg_H(v'_1) = deg_H(v'_2) = \cdots = deg_H(v'_{k-t-1}) = 2$, and every endvertex of H has distance $\le t'$ either to v'_0 or to $v'_{k-t'}$. Since H does not contain cycles, H is a $P^*_{k,t'}$, $0 \le t \le k-2$. As $R' \nsubseteq B$, we have $B \notin P^*_{k,t'}$.

(iii) Finally, suppose that G does not contain a $P^*_{k,t}$, $0 \le t \le k-2$. We show that $P_k(G)$ is a connected graph.

Let $a, b \in V(P_k(G))$. First suppose that $A \cap B$ does not contain an edge. Let $P = (y_0, y_1, \ldots, y_l)$ be a shortest path in G joining a vertex of A with a vertex of B (i.e., $y_l \in V(B)$). By Lemma 2, there is a $b' \in Co(b)$ such that y_l is an endvertex of B' and the edge of B' incident with y_l lies in B. Let $B' = (b'_0, b'_1, \ldots, b'_{k-1}, y_l)$. Then $P' = (b'_0, b'_1, \ldots, b'_{k-1}, y_l, y_{l-1}, \ldots, y_0)$ is a walk of length $k + l$. Since G does not contain a cycle of length $\le k$, every subwalk of P' of length k is a path. Let B'' be a subpath of length k of P' with endvertex y_0. Then $d_{P_k(G)}(b', b'') \le l$, and hence $b'' \in Co(b)$. As (A, B'') forms a T in G, we have $b \in Co(a)$, by Lemma 3.

Now suppose that $A \cap B$ contains an edge. Let $P = (y_0, y_1, \ldots, y_l)$ be a longest path that is shared by A and B. By Lemma 2, for every i, $0 \le i \le l$, there is a $b_i \in Co(b)$ such that y_i is an endvertex of B_i, and the edge of B_i incident with y_i lies in B. If B_0 does not contain the edge y_0y_1, then (A, B_0) forms a T in G, so that $b \in Co(a)$, by Lemma 3. Analogously, if B_l does not contain $y_{l-1}y_l$, then $b \in Co(a)$. Thus, suppose that B_0 contains the edge y_0y_1 and B_l contains $y_{l-1}y_l$. Then there is some i, $0 \le i < l$, such that both B_i and B_{i+1} contain the edge y_iy_{i+1}. By Lemma 2, there is an $a' \in Co(a)$ such that y_i is an endvertex of A' and the edge of A' incident with y_i lies in A. If A' contains the edge y_iy_{i+1}, then

$d_{P_k(G)}(a', b_{i+1}) \leq k-1$, and hence $b \in Co(a)$. On the other hand, if A' does not contain $y_i y_{i+1}$, we have $d_{P_k(G)}(a', b_i) \leq k$, and hence $b \in Co(a)$ as well. □

3. P_3-PATH GRAPHS

Let G be a graph and let A be a path of length three in G. By P_3° we denote a subgraph of G induced by vertices of a path of length 3, say (v_0, v_1, v_2, v_3), such that neither v_0 nor v_3 has a neighbour in $V(G) - \{v_1, v_2\}$. We say that the path A is in P_3°, $A \in P_3^\circ$, if $A = (v_0, v_1, v_2, v_3)$.

By P_4° we denote an induced subgraph of G with a path (x, v_0, v_1, v_2, y), in which every neighbour of v_0 (and analogously every neighbour of v_2), except v_0, v_1 and v_2, has degree 1, or it has degree 2 and in this case it is adjacent to v_1. Moreover, no vertex of $V(P_4^\circ) - \{v_1\}$ is joined by an edge to a vertex of $V(G) - V(P_4^\circ)$ in G. The path (v_0, v_1, v_2) is a **base** of P_4°, and we say that the path A lies in P_4°, $A \in P_4^\circ$, if the base of P_4° is a subpath of A.

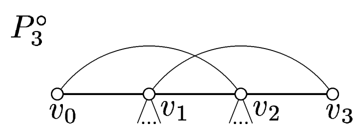

Figure 2

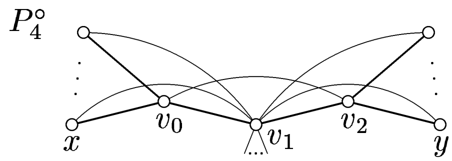

Figure 3

In Figure 2 a P_3° is pictured and a P_4° is in Figure 3. The edges that must be in G are painted thick, while edges, that are not necessarily in G, are painted thin.

Let K_4 be a complete graph on 4 vertices, and let S be a set (possibly empty) of independent vertices. A graph obtained from $K_4 \cup S$ by joining all vertices of S to one special vertex of K_4 is denoted by K_4^*, see Figure 4. Let $K_{2,t}$ be a complete bipartite graph, $t \geq 1$, and let (X, Y) be the bipartition of $K_{2,t}$, $X = \{v_1, v_2\}$. Join t sets of independent vertices by edges, each to one vertex of Y; further, glue a set of stars (each with at least 3 vertices) by one endvertex, each either to v_1 or to v_2; glue a set of triangles by one vertex, each either to v_1 or to v_2; and finally, join v_1 to v_2 by an edge. The resulting graph is denoted by $K_{2,t}^*$, see Figure 5.

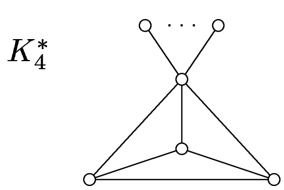

Figure 4

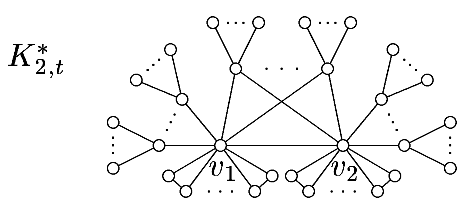

Figure 5

Theorem 4. *Let G be a connected graph such that $P_3(G)$ is not empty. Then $P_3(G)$ is disconnected if and only if one of the following holds:*

(1) *G contains a P_t°, $t \in \{3, 4\}$, and a path A of length 3 such that $A \notin P_t^\circ$;*
(2) *G is isomorphic to K_4^*;*
(3) *G is isomorphic to $K_{2,t}^*$, $t \geq 1$.*

If $A \in P_3^\circ$ in G, then a is an isolated vertex in $P_3(G)$, and if $A \in P_4^\circ$, then a lies in a complete bipartite graph. Thus, we have the following corollary of Theorem 4.

Corollary 5. *Let G be a connected graph that is not isomorphic to K_4^* or to $K_{2,t}^*$, $t \geq 1$. Then at most one nontrivial component of $P_3(G)$ is different from a complete bipartite graph.*

In the proof of Theorem 4 we use 6 lemmas.

Lemma 6. *Let G be a connected graph, and let a and b be vertices in $P_3(G)$. If neither A nor B is in some P_3° or P_4° in G, then there are vertices c and d in $P_3(G)$, such that $c \in Co(a)$, $d \in Co(b)$ and C and D share an edge in G.*

Proof. Let $A \cap B$ do not contain an edge, and let $P = (y_0, y_1, \ldots, y_l)$ be a shortest path in G joining a vertex of A with a vertex of B (i.e., $y_l \in V(B)$). We show that there is a vertex b' in $Co(b)$, such that y_l is an endvertex of B'.

Suppose that there is no vertex b' with the required property. Then $B = (x_0, x_1, y_l, x_3)$, and since B is not in a P_3° in G, there is a vertex $\overline{b}$ in $P_3(G)$ such that $\overline{b}b \in E(P_3(G))$. By our assumption, $\overline{B} = (x_1, y_l, x_3, x_4)$ for some $x_4 \in V(G)$. Moreover, for every neighbour u of b we have $U = (x_1, y_l, x_3, z)$, where z has no neighbours in $V(G) - \{y_l, x_3\}$; and for every neighbour v of $\overline{b}$ we have $V = (z, x_1, y_l, x_3)$, where z has no neighbours in $V(G) - \{x_1, y_l\}$. Hence B is in a P_4°, a contradiction.

Thus, there is a vertex $b' \in Co(b)$, such that y_l is an endvertex of B'. Let b'' be the first vertex on a shortest $b - b'$ path in $P_3(G)$, such that one endvertex of B'' is in P. Assume that $B'' = (b''_3, b''_2, b''_1, y_i)$. Then $P' = (b''_3, b''_2, b''_1, y_i, y_{i-1}, \ldots, y_0)$ is a path of length $i+3 \geq 3$. Let B^* be a subpath of P of length 3, such that y_0 is an endvertex of B^*. Then $d_{P_3(G)}(b'', b^*) = i$, and hence, $b^* \in Co(b)$.

Denote $B^* = (y_0, b_1^*, b_2^*, b_3^*)$, and suppose that $A \cap B^*$ does not contain an edge. Let $A = (a_0, a_1, a_2, a_3)$. Distinguish two cases.

(i) $y_0 = a_1$. Then $b_1^* \neq a_0$ and $b_1^* \neq a_2$, so that at least one of a_0 and a_2, say a_0, is different from b_2^*. Since a_0 is not an interior vertex of B^*, $D = (a_0, y_l, b_1^*, b_2^*)$ is a path of length 3 in G. As $b^*d \in E(P_3(G))$, we have $d \in Co(b)$ and $A \cap D$ contains an edge.

(ii) $y_0 = a_0$. If $b_1^* \neq a_2$ then $C = (b_1^*, y_0, a_1, a_2)$ is a path of length 3 in G, $c \in Co(a)$, $b^* \in Co(b)$, and $C \cap B^*$ contains an edge. On the other hand, if $b_1^* = a_2$ then $D = (a_1, y_0, a_2, b_2^*)$ is a path of length 3 in G, $d \in Co(b)$, and $A \cap D$ contains an edge. □

Lemma 7. *Let G be a connected graph, and let a and b be two vertices in $P_3(G)$ such that $b \notin Co(a)$ and $A \cap B$ contains a path of length two. Moreover, suppose G does not contain a P_3° or a P_4°. Then G is isomorphic either to K_4^* or to $K_{2,t}^*$ for some $t \geq 1$.*

Proof. Let $A = (x_0, x_1, x_2, x_3)$ and $B = (x_0, x_1, x_2, x_4)$, $x_3 \neq x_4$. Since $b \notin Co(a)$, x_0 has no neighbour in $V(G) - \{x_1, x_2\}$. Thus, both x_3 and x_4 have some neighbours in $V(G) - \{x_1, x_2\}$, as G does dot contain a P_3°. Let y be a vertex of G such that $x_1 y \in E(G)$ and $y \notin \{x_0, x_2, x_3, x_4\}$. Then $a' \in Co(a)$ and $b' \in Co(b)$, where $A' = (y, x_1, x_2, x_3)$ and $B' = (y, x_1, x_2, x_4)$. Since $b \notin Co(a)$ we have $b' \notin Co(a')$, and hence, y has no neighbour in $V(G) - \{x_1, x_2\}$.

Suppose that $x_3 x_4 \in E(G)$ and distinguish three cases.

Case (1): $x_1 x_3, x_1 x_4 \in E(G)$, see Figure 6.

Let G' be a graph obtained from G by joining x_0 to x_2. Then A, (x_1, x_2, x_3, x_4), (x_2, x_3, x_4, x_1), (x_3, x_4, x_1, x_0), (x_4, x_1, x_0, x_2), (x_1, x_0, x_2, x_4), (x_0, x_2, x_4, x_3), (x_2, x_4, x_3, x_1), (x_1, x_2, x_4, x_3), B is a sequence of paths whose images produce a walk of length 9 from a to b in $P_3(G')$. (We remark that $d_{P_3(G')}(a,b) = 9$.) Thus $b \in Co(a)$, a contradiction. Hence $deg_G(x_0) = 1$.

Let $C_1 = (x_1, x_2, x_3, x_4)$ and $C_2 = (x_1, x_2, x_4, x_3)$ be two cycles of length 4 in G. For every subpath A' of C_1 of length 3 we have $a' \in Co(a)$, and for every subpath B' of C_2 of length 3 we have $b' \in Co(b)$. Let y be a vertex in $V(G) - \{x_1, \ldots, x_4\}$ which is joined to some $x \in \{x_1, \ldots, x_4\}$. Since $C_1 \cap C_2$ contains an edge incident with x, there are paths A'' and B'' of length 3 in G, both containing the edge yx, such that $a'' \in Co(a)$, $b'' \in Co(b)$ and $A'' \cap B''$ contains a P_2. Thus, analogously as above it can be shown that $deg_G(y) = 1$. Finally, as G does not contain a P_3° we have $x = x_1$, and hence $G \cong K_4^*$.

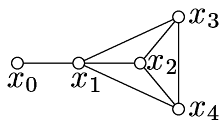

Figure 6

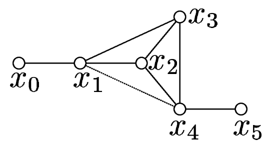

Figure 7

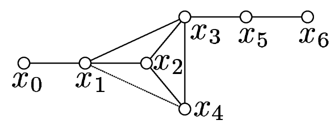

Figure 8

Case (2): $x_1x_3 \in E(G)$ and $x_1x_4 \notin E(G)$, see Figure 7 and Figure 8 (by dotted lines edges that are missing in G are pictured).

Since (x_1, x_2, x_3) is not a base of P_4°, either there is a vertex $y \in V(G) - \{x_0, \ldots, x_4\}$ such that $yx_4 \in E(G)$, or there is a path of length 2 glued by one endvertex to x_3 (the other vertices of the path are not in $\{x_0, \ldots, x_4\}$).

First suppose that there is an $x_5 \in V(G) - \{x_0, \ldots, x_4\}$ such that $x_4x_5 \in E(G)$, see Figure 7. Let G' be a graph obtained from G by joining x_0 to x_2. Then A, (x_1, x_2, x_3, x_4), (x_2, x_3, x_4, x_5), (x_0, x_2, x_3, x_4), (x_1, x_0, x_2, x_3), (x_3, x_1, x_0, x_2), (x_4, x_3, x_1, x_0), (x_2, x_4, x_3, x_1), (x_1, x_2, x_4, x_3), B is a sequence of paths whose images produce a walk of length 9 from a to b in $P_3(G')$. Thus $b \in Co(a)$, a contradiction.

Hence $deg_G(x_0) = 1$. Analogously, for every vertex x, such that $xx_2, xx_3 \in E(G)$, every neighbour of x (different from x_2 and x_3) has degree 1 in G.

Let y_1 and y_2 be vertices in $V(G) - \{x_0, \ldots, x_5\}$, such that $x_2y_1, y_1y_2 \in E(G)$. If y_2 is joined by an edge to a vertex, say z, of $V(G) - \{x_2, y_1\}$, then for $C = (x_2, y_1, y_2, z)$ we have $c \in Co(a)$ and $c \in Co(b)$. Hence $b \in Co(a)$, a contradiction. Since G contains a P_3° if there is a vertex of degree 1 joined to x_2, we have $G \cong K_{2,t}^*$ for some $t \geq 2$.

Now suppose that there are $x_5, x_6 \in V(G) - \{x_0, \ldots, x_4\}$ such that $x_3x_5, x_5x_6 \in E(G)$, see Figure 8. (Observe that the cases $x_6 \in \{x_0, x_1, x_4\}$ imply $b \in Co(a)$.)

Let G' be a graph obtained from G by joining x_0 to x_2. Then A, (x_1, x_2, x_3, x_5), (x_2, x_3, x_5, x_6), (x_0, x_2, x_3, x_5), (x_1, x_0, x_2, x_3), (x_3, x_1, x_0, x_2), (x_4, x_3, x_1, x_0), (x_2, x_4, x_3, x_1), (x_1, x_2, x_4, x_3), B is a sequence of paths whose images produce a walk of length 9 from a to b in $P_3(G')$. Thus $b \in Co(a)$, a contradiction.

Hence $deg_G(x_0) = 1$. Analogously, for every vertex x, such that $xx_2, xx_3 \in E(G)$, every neighbour of x (different from x_2 and x_3) has degree 1 in G. Now analogously as above it can be shown that $G \cong K_{2,t}^*$ for some $t \geq 2$.

Case (3): $x_1x_3, x_1x_4 \notin E(G)$, see Figure 9.

Since neither (x_1, x_2, x_3) nor (x_1, x_2, x_4) is a base of P_4°, there is a vertex $x_5 \in V(G) - \{x_0, \ldots, x_4\}$ which is adjacent either to x_3 or to x_4. Assume that $x_3x_5 \in E(G)$. As $b \notin Co(a)$, x_5 has no neighbour in $\{x_0, x_1, x_4\}$. Since (x_1, x_2, x_3) is not a base of P_4°, there is a vertex $y \in V(G) - \{x_0, \ldots, x_5\}$ such that either $yx_5 \in E(G)$

or $yx_4 \in E(G)$.

First suppose that there is an $x_6 \in V(G) - \{x_0, \ldots, x_5\}$ such that $x_5x_6 \in E(G)$. Then every neighbour of x_4 (different from x_2 and x_3) has degree 1 in G, otherwise $b \in Co(a)$. Analogously, for every vertex x, such that $xx_2, xx_3 \in E(G)$, every neighbour of x (different from x_2 and x_3) has degree 1 in G. Thus, analogously as above we have $G \cong K^*_{2,t}$ for some $t \geq 1$.

If there is an $x_6 \in V(G) - \{x_0, \ldots, x_5\}$ such that $x_4x_6 \in E(G)$, then the problem is reduced to the previous case as (x_3, x_2, x_4) is not a base of P_4°.

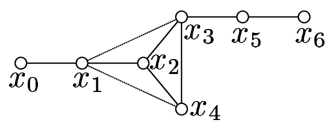

Figure 9

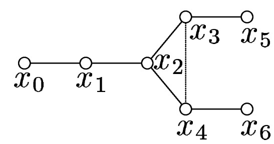

Figure 10

To prove the lemma it remains to consider the case $x_3x_4 \notin E(G)$, see Figure 10.

As $b \notin Co(a)$, there is no cycle $(x_3, x_2, x_4, \ldots)$ of length at least 4 in G. Since neither A nor B is in a P_3° in G, there are $x_5, x_6 \in V(G) - \{x_0, \ldots, x_5\}$, $x_5 \neq x_6$, such that $x_3x_5, x_4x_6 \in E(G)$. Moreover, as G does not contain a P_4° with base (x_1, x_2, x_3), there is an $x_7 \in V(G) - \{x_0, \ldots, x_6\}$ such that $x_5x_7 \in E(G)$, and analogously, there is an $x_8 \in V(G) - \{x_0, \ldots, x_7\}$ such that $x_6x_8 \in E(G)$. (Observe that $b \in Co(a)$ if $x_7 = x_1$, and the same holds if $x_8 = x_1$.) But now $d_{P_3(G)}(a, b) \leq 7$, and hence $b \in Co(a)$, a contradiction. □

Lemma 8. *Let G be a connected graph, and let a and b be two vertices in $P_3(G)$ such that $b \notin Co(a)$ and $A \cap B$ contains two independent edges. Moreover, suppose G does not contain a P_3° or a P_4°. Then G is isomorphic either to K_4^* or to $K^*_{2,t}$ for some $t \geq 1$, or there are $a' \in Co(a)$ and $b' \in Co(b)$ such that $A' \cap B'$ contains a path of length 2.*

Proof. Let $A = (x_0, x_1, x_2, x_3)$. Since $b \notin Co(a)$, $B = (x_0, x_1, x_3, x_2)$. We may assume that x_0 has no neighbour in $V(G) - \{x_0, \ldots, x_3\}$, as otherwise there are $a' \in Co(a)$ and $b' \in Co(b)$ such that $A' \cap B'$ contains a P_2.

Distinguish three cases.

Case 1: $x_0x_2, x_0x_3 \in E(G)$. Then both A and B lie in cycles of length 4. If there is a vertex y adjacent to a vertex of $\{x_0, \ldots, x_4\}$, then there are $a' \in Co(a)$ and $b' \in Co(b)$ such that $A' \cap B'$ contains a P_2. Thus, $G \cong K_4$ which is a special K_4^*.

Case 2: $x_0x_2 \in E(G)$ and $x_0x_3 \notin E(G)$, see Figure 11. Since A is not in a P_3° in G, there is an $x_4 \in V(G) - \{x_0, \ldots, x_3\}$ such that $x_3x_4 \in E(G)$. But then $a' \in Co(a)$, $b' \in Co(b)$ and $A' \cap B'$ contains a P_2, where $A' = (x_1, x_2, x_3, x_4)$ and $B' = (x_0, x_2, x_3, x_4)$.

Case 3: $x_0x_2, x_0x_3 \notin E(G)$, see Figure 12. Since neither A nor B is in a P_3° in G, there are vertices $x_4, x_5 \in V(G) - \{x_0, \ldots, x_3\}$ such that $x_2x_4, x_3x_5 \in E(G)$. We may assume that the degree of every neighbour of x_1 (except x_2 and x_3) is 1 in G, as the other possibilities we have already solved.

If $x_4 \neq x_5$, then there are $x_6, x_7 \in V(G) - \{x_0, \ldots, x_3\}$ such that $x_4x_6, x_5x_7 \in E(G)$, as neither (x_1, x_3, x_2) nor (x_1, x_2, x_3) is a base of P_4°. But then $b \in Co(a)$, a contradiction.

Thus, suppose that $x_4 = x_5$. By previous subcase, we may assume that $deg_G(x_2) = deg_G(x_3) = 3$. As (x_1, x_2, x_3) is not a base of P_4°, there is an $x_5 \in V(G) - \{x_0, \ldots, x_4\}$ such that $x_4x_5 \in E(G)$. By our assumptions, $deg_G(x_5) = 1$. Hence, $deg_G(x_0) = deg_G(x_5) = 1$, $deg_G(x_2) = deg_G(x_3) = 3$, and all neighbours of x_1 and x_4 (except x_2 and x_3) have degree 1 in G. Thus, $G \cong K_{2,2}^*$. □

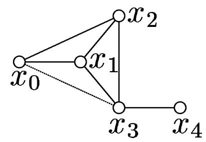

Figure 11

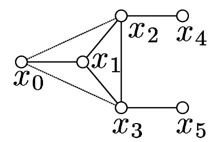

Figure 12

Lemma 9. *Let G be a connected graph, and let a and b be two vertices in $P_3(G)$ such that $b \notin Co(a)$ and $A \cap B$ contains exactly one edge and two vertices outside this edge. Moreover, suppose G does not contain a P_3° or a P_4°. Then there are $a' \in Co(a)$ and $b' \in Co(b)$ such that $A' \cap B'$ contains two independent edges.*

Proof. Let $A = (x_0, x_1, x_2, x_3)$. Then either $B = (x_0, x_2, x_1, x_3)$ or $B = (x_1, x_2, x_0, x_3)$.

First suppose that $B = (x_0, x_2, x_1, x_3)$. Since A is not in a P_3° in G, either $x_0x_3 \in E(G)$ or $x_3x_4 \in E(G)$ for some $x_4 \in V(G) - \{x_0, \ldots, x_3\}$. In both these cases there are $a' \in Co(a)$ and $b' \in Co(b)$ such that $A' \cap B'$ contains two independent edges.

Now suppose that $B = (x_1, x_2, x_0, x_3)$. Then for $A' = (x_1, x_2, x_3, x_0)$ we have $a' \in Co(a)$, and $A' \cap B$ contains two independent edges. □

Lemma 10. *Let G be a connected graph, and let a and b be two vertices in $P_3(G)$ such that $b \notin Co(a)$ and $A \cap B$ contains exactly one edge and one vertex outside this edge. Moreover, suppose G does not contain a P_3° or a P_4°. Then there are $a' \in Co(a)$ and $b' \in Co(b)$ such that $A' \cap B'$ contains two edges.*

Proof. Let $A = (x_0, x_1, x_2, x_3)$, and let x_4 be a vertex of B lying outside A. Distinguish four cases.

Case 1: Suppose that x_1x_2 is the middle edge of B. Then $B = (x_3, x_1, x_2, x_4)$. If x_4 has a neighbour in $V(G) - \{x_1, x_2\}$, then for $B' = (x_0, x_1, x_2, x_4)$ we have $b' \in Co(b)$ and $A \cap B' = P_2$. Thus, we may assume that both x_0 and x_4 have no neighbour in $V(G) - \{x_1, x_2\}$. However, then there is a P_3° in G, a contradiction.

Case 2: Suppose that x_1x_2 is an endedge of B.

If $B = (x_1, x_2, x_0, x_4)$, then for $A' = (x_4, x_0, x_1, x_2)$ we have $a' \in Co(a)$ and $A' \cap B$ contains two independent edges.

If $B = (x_1, x_2, x_4, x_0)$ then $b \in Co(a)$; and if $B = (x_1, x_2, x_4, x_3)$, then for $B' = (x_0, x_1, x_2, x_4)$ we have $b' \in Co(b)$ and $A \cap B' = P_2$.

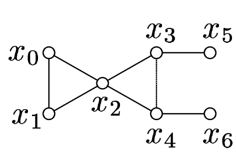

Figure 13

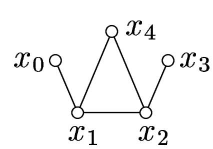

Figure 14

Case 3: Suppose that x_0x_1 is an endedge of B and x_1 is an endvertex of B. If $B = (x_1, x_0, x_4, x_2)$, $B = (x_1, x_0, x_3, x_4)$, or $B = (x_1, x_0, x_4, x_3)$, then $b \in Co(a)$. Thus, suppose that $B = (x_1, x_0, x_2, x_4)$, see Figure 13.

If $deg_G(x_1) > 2$, then for $B' = (x_1, x_0, x_2, x_3)$ we have $b' \in Co(b)$ and $A \cap B'$ contains two independent edges. Thus, suppose that $deg_G(x_0) = deg_G(x_1) = 2$.

If $x_3x_4 \in E(G)$, then analogously as above we have $deg_G(x_3) = deg_G(x_4) = 2$, and hence, there is a P_4° with base (x_0, x_2, x_3) in G, a contradiction. Thus, suppose that $x_3x_4 \notin E(G)$.

As $b \notin Co(a)$, there is no cycle $(x_3, x_2, x_4, \dots)$ of length at least 4 in G. Since neither A nor B is in a P_3° in G, there are $x_5, x_6 \in V(G) - \{x_0, \dots, x_4\}$, $x_5 \neq x_6$, such that $x_3x_5, x_4x_6 \in E(G)$. Moreover, as G does not contain a P_4° with base (x_0, x_2, x_3), there is an $x_7 \in V(G) - \{x_0, \dots, x_6\}$ such that $x_5x_7 \in E(G)$. Thus, for $A' = (x_6, x_4, x_2, x_3)$ and $B' = (x_0, x_2, x_4, x_6)$ we have $a' \in Co(a)$, $b' \in Co(b)$ and $A' \cap B' = P_2$.

Case 4: Suppose that x_0x_1 is an endedge of B and x_0 is an endvertex of B.

If $B = (x_0, x_1, x_4, x_3)$, then $b \in Co(a)$. Since the cases $B = (x_0, x_1, x_4, x_2)$ and $B = (x_0, x_1, x_3, x_4)$ are equivalent, suppose that $B = (x_0, x_1, x_4, x_2)$, see Figure 14.

We have $x_0x_3 \notin E(G)$, as otherwise $b \in Co(a)$. Since A is not in a P_3° in G, there is a $y \in V(G) - \{x_0, \dots, x_3\}$ such that either $x_0y \in E(G)$ or $x_3y \in E(G)$. Assume that $x_0y \in E(G)$. If $y \neq x_4$, then for $A' = (y, x_0, x_1, x_2)$ and $B' = (y, x_0, x_1, x_4)$ we have $a' \in Co(a)$, $b' \in Co(b)$ and $A' \cap B' = P_2$. On the other hand, if $y = x_4$, then for $A' = (x_2, x_4, x_0, x_1)$ we have $a' \in Co(a)$ and $A' \cap B$ contains two independent edges. $\square$

Lemma 11. *Let G be a connected graph, and let a and b be two vertices in $P_3(G)$ such that $b \notin Co(a)$ and $A \cap B$ contains exactly one edge and no vertex outside this edge. Moreover, suppose G does not contain a P_3° or a P_4°. Then there are $a' \in Co(a)$ and $b' \in Co(b)$ such that $A' \cap B'$ contains two edges.*

Proof. Let $A = (x_0, x_1, x_2, x_3)$, and let x_4 and x_5 be vertices of B lying outside A. If $A' \cap B'$ does not contain a P_2 for every $a' \in Co(a)$ and $b' \in Co(b)$, then either $B = (x_0, x_1, x_4, x_5)$ or $B = (x_4, x_1, x_2, x_5)$.

First suppose that $B = (x_0, x_1, x_4, x_5)$, see Figure 15. If there is a $y \in V(G) - \{x_1, x_2\}$ such that $yx_3 \in E(G)$, then for $A' = (x_5, x_4, x_1, x_2)$ we have $a' \in Co(a)$ and $A' \cap B = P_2$. Hence, we may assume that x_3 has no neighbour in $V(G) - \{x_1, x_2\}$. Since A is not in a P_3° in G, there is a $y \in V(G) - \{x_1, x_2\}$ such that $yx_0 \in E(G)$. If $y \neq x_4$, then for $A' = (y, x_0, x_1, x_2)$ and $B' = (y, x_0, x_1, x_4)$ we have $a' \in Co(a)$, $b' \in Co(b)$ and $A' \cap B' = P_2$. On the other hand, if $x_0x_4 \in E(G)$, then for $A' = (x_5, x_4, x_0, x_1)$ we have $a' \in Co(a)$ and $A' \cap B$ contains two edges.

Thus, suppose that $B = (x_4, x_1, x_2, x_5)$. Since A is not in a P_3° in G, we may assume that there is a $y \in V(G) - \{x_1, x_2\}$ such that $x_0y \in E(G)$. Then for $A' = (x_0, x_1, x_2, x_5)$ we have $a' \in Co(a)$ and $A' \cap B = P_2$. $\square$

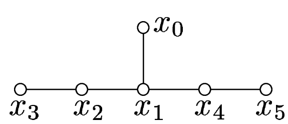

Figure 15

Now we prove Theorem 4.

Proof of Theorem 4. First suppose that G contains a P_3° and a path A of length 3 such that $A \notin P_3^\circ$. Then there is a path B of length 3 in G such that $B \in P_3^\circ$. Since b is an isolated vertex in $P_3(G)$, $b \notin Co(a)$. Now suppose that G contains a P_4°, and choose $B \in P_4^\circ$. For every vertex $b' \in Co(b)$, the B' contains the base of P_4°. Hence, $P_3(G)$ is disconnected if there is a path A of length 3 such that $A \notin P_4^\circ$.

If G is isomorphic to K_4^*, then $P_3(G)$ has three components, each containing a C_4. Finally, if G is isomorphic to $K_{2,t}^*$, $t \geq 1$, and $P_3(G)$ is not empty, then some paths of length 3 in G contain the edge v_1v_2, while the other do not, see Figure 5. Let $a \in V(P_3(G))$ such that $v_1v_2 \in A$. Then $v_1v_2 \in A'$ for every $a' \in Co(a)$, so that $P_3(G)$ is a disconnected graph.

To prove the "only if" part of Theorem 4, first suppose that G contains a P_t°, $t \in \{3,4\}$, but no path A of length 3 such that $A \notin P_t^\circ$. If G contains a P_3°, then our assumption implies that G is a path of length 3. On the other hand, if G contains a P_4° and there is no P_3° in G, then G is a tree of diameter 4 and $P_3(G)$ is a complete bipartite graph. Thus, in what follows we restrict our considerations to graphs which do not contain a P_t°, $t \in \{3,4\}$.

Let G be a graph which does not contain a P_3° or a P_4°, and let a and b be vertices of $P_3(G)$ such that $b \notin Co(a)$. By Lemma 6, there are $a' \in Co(a)$ and $b' \in Co(b)$ such that $A' \cap B'$ contains an edge. Hence, G is either isomorphic to K_4^* or to $K_{2,t}^*$, $t \geq 1$, by Lemmas 7, 8, 9, 10 and 11. □

Acknowledgment

The authors sincerely acknowledge the helpful remarks and corrections of the referee.

References

[1] Belan A., Jurica P., *Diameter in path graphs*, Acta Math. Univ. Comenian. **LXVIII** (1999), 111-126.

[2] Broersma H. J., Hoede C., *Path graphs*, J. Graph Theory **13** (1989), 427-444.

[3] Knor M., Niepel Ľ., *Path, trail and walk graphs*, Acta Math. Univ. Comenian. **LXVIII** (1999), 253-256.

[4] Knor M., Niepel Ľ., *Distances in iterated path graphs*, Discrete Math. (to appear).

[5] Knor M., Niepel Ľ., *Centers in path graphs*, (submitted).

[6] Knor M., Niepel Ľ., *Graphs isomorphic to their path graphs*, (submitted).

[7] Li H., Lin Y., *On the characterization of path graphs*, J. Graph Theory **17** (1993), 463-466.

[8] Li X., Zhao B., *Isomorphisms of P_4-graphs*, Australasian J. of Combinatorics **15** (1997), 135-143.

[9] Yu X., *Trees and unicyclic graphs with Hamiltonian path graphs*, J. Graph Theory **14** (1990), 705-708.