PATH, TRAIL AND WALK GRAPHS

MARTIN KNOR[1)] AND ĽUDOVÍT NIEPEL[2)]

[1)] Slovak University of Technology, Faculty of
Civil Engineering, Department of Mathematics,
Radlinského 11, 813 68 Bratislava, Slovakia;

[2)] Kuwait University, Faculty of Science,
Department of Mathematics & Computer
Science, P.O. box 5969 Safat 13060, Kuwait.

ABSTRACT. We introduce trail graphs and walk graphs as a generalization of line graphs. The path graph $P_k(G)$ is an induced subgraph of the trail graph $T_k(G)$, which is an induced subgraph of the walk graph $W_k(G)$. We prove that the walk graph $W_k(G)$ is an induced subgraph of the k-iterated line graph $L^k(G)$, using a special embedding preserving histories. Hence, trail graphs and walk graphs are in a sense more close to line graphs than the path graphs, and some problems that are complicated in path graphs become easier for walk graphs.

1. INTRODUCTION

Let G be a graph and $k \geq 1$. Let $\mathcal{P}_k$ be the set of paths of length k in G, let $\mathcal{T}_k$ be the set of trails of length k in G, and let $\mathcal{W}_k$ be the set of walks of length k in G in which no two consecutive edges are equal. The vertex set of the **path graph** $P_k(G)$ (**trail graph** $T_k(G)$ and **walk graph** $W_k(G)$) is the set $\mathcal{P}_k$ ($\mathcal{T}_k$ and $\mathcal{W}_k$). Two vertices of $P_k(G)$ ($T_k(G)$ and $W_k(G)$) are joined by an edge if and only if one of the corresponding walks can be obtained from the other by deleting an edge from one end and adding an edge to the other end. It means that the vertices are adjacent if and only if one can be obtained from the other by "shifting" the corresponding walks in G.

Path graphs were investigated by Broersma and Hoede in [2] (see also [1], [4], [6] and [8]), as a natural generalization of line graphs (observe that $P_1(G)$ is the line graph of G, i.e., $P_1(G) = L(G)$). Although the relation $P_1(G) = L(G)$ is the

1991 *Mathematics Subject Classification.* 05C38.
Key words and phrases. Iterated line graph, line graph, path graph, trail graph, walk graph.
[1)] Supported by VEGA grant
[2)] Supported by Kuwait University grant #SM 172

unique motivation for studying path graphs presented in [2], in this note we show that there is a stronger connection of path graphs to line graphs.

The i-iterated line graph of G, the $L^i(G)$, is defined as

$$L^i(G) = \begin{cases} G & \text{if } i = 0; \\ L(L^{i-1}(G)) & \text{if } i > 0. \end{cases}$$

We prove that there is a special embedding of $P_k(G)$ in $L^k(G)$, which implies that the path graph $P_k(G)$ is a subgraph of $L^k(G)$. This yields another connection of path graphs to line graphs. In fact, we prove more. We prove that the walk graph $W_k(G)$ is a subgraph of $L^k(G)$. As $P_k(G)$ is an induced subgraph of the trail graph $T_k(G)$, and $T_k(G)$ is an induced subgraph of $W_k(G)$, we have:

$$\begin{aligned} P_1(G) &= T_1(G) = W_1(G) = L(G) \\ P_2(G) &= T_2(G) = W_2(G) \subseteq L^2(G) \\ P_3(G) &\leq T_3(G) = W_3(G) \subseteq L^3(G) \\ P_k(G) &\leq T_k(G) \leq W_k(G) \subseteq L^k(G) \quad \text{if } k \geq 4. \end{aligned} \tag{*}$$

(We remark that $\leq$ is used for an induced subgraph, while $\subseteq$ denotes a subgraph.) Observe that $W_2(G)$ is a spanning subgraph of $L^2(G)$. By (*) trail graphs and walk graphs are even more close to line graphs than the path graphs.

When studying P_k-path graphs, cycles of length smaller than $k+1$ make real obstacles. For instance, the statement of Lemma 3 in [1] is restricted to graphs which do not contain such small cycles if $k \geq 5$. This causes also a restriction of the main theorem in [1]. An analogous situation appears in [4], where the authors prove a necessary and sufficient condition for a graph G to have a connected P_k-path graph, provided that G does not contain small cycles. This is not the case of walk graphs, where small cycles cannot cause any problem. Hence, some difficult problems stated for path graphs, e.g. the connectivity, determining the distances, maybe also Hamiltonicity, become easier for walk graphs. This all seems to provide enough motivation for a further study of walk graphs.

2. The result

In what follows we consider only walks in which no two consecutive edges are equal. For easier handling of walks of length k in G (i.e., the vertices of $W_k(G)$) we adopt the following convention. We denote the vertices of $W_k(G)$ (as well as the vertices of G) by small letters a, b, ..., while the corresponding walks of length k in G will be denoted by capital letters A, B, ... It means that if A is a walk of length k in G and a is a vertex in $W_k(G)$, then a must be the vertex corresponding to the walk A.

We now introduce the concept of history.

Let G be a graph, $i \geq 0$, and let v be a vertex in $L^i(G)$.

1° The **0-history** $B^0(v)$ is a subgraph of $L^i(G)$ formed by the unique vertex v.

2° If $0 < j \leq i$, the **j-history** $B^j(v)$ is a subgraph of $L^{i-j}(G)$. Assume that $V(B^{j-1}(v)) = \{a_1, a_2, \ldots, a_l\}$. Then, the vertices and edges of $B^j(v)$ are the vertices and edges of $A_1, A_2, \ldots, A_l$.

Recall that $L(G) = W_1(G)$, and hence, according to our agreement $A_1, A_2, \dots, A_l$ are edges corresponding to the vertices $a_1, a_2, \dots, a_l$ in the preceding definition.

Let v be a vertex in $L^i(G)$. Then $B^i(v)$ is the minimum subgraph of G such that $L^i(B^i(v))$ contains the vertex v. Thus, one can imagine the history $B^i(v)$ as a "footprint" of v in G.

We have to point out that the notion of history is crucial in determining the distances in iterated line graphs and P_2-path graphs.

In [7] it is proved that every connected subgraph of G with at most i edges, different from a path with fewer than i edges, is an i-history of a vertex in $L^k(G)$. If u and v are vertices in $L^i(G)$, then their distance can be computed already in G (if it is large enough), using i-histories. More precisely, in [7] it is proved that

$$d_{L^i(G)}(u, v) = d_{L^{i-j}(G)}(B^j(u), B^j(v)) + j$$

if the j-histories $B^j(u)$ and $B^j(v)$ are edge-disjoint (see also [5] for an analogous result in iterated P_2-path graphs). This result was used for determining the behavior of the diameter and the radius of i-iterated line graph (as a function of i), see [7], and for a theorem that states that almost all i-iterated line graphs are selfcentric of diameter (and radius) $i+2$, see [3].

Let U be a walk of length k in G. By $\tilde{U}$ we denote the subgraph of G formed by the vertices and edges of U.

We are now ready to prove the main result of this note.

Theorem 1. *Let G be a graph and $k \geq 2$. Then there is an embedding $\varphi : W_k(G) \to L^k(G)$ such that for each vertex u in $W_k(G)$, the walk U and the k-history $B^k(\varphi(u))$ satisfy $\tilde{U} = B^k(\varphi(u))$.*

Proof. Let U be a walk of length i in G, $U = (u_0, u_1, \dots, u_i)$, and let $E_0{=}(u_0, u_1)$, $E_1{=}(u_1, u_2)$, ..., $E_{i-1}{=}(u_{i-1}, u_i)$ be edges of U. Then $e_0, e_1 \dots, e_{i-1}$ are vertices in $L(G)$ and $K(U) = (e_0, e_1, \dots, e_{i-1})$ is a walk of length $i{-}1$ in $L(G)$. (Observe that no two consecutive edges of $K(U)$ are equal.) Define

$$K^i(U) = \begin{cases} U & \text{if } i = 0; \\ K(K^{i-1}(U)) & \text{if } i > 0. \end{cases}$$

Then $K^i(U)$ is a single vertex in $L^i(G)$.

Let $\varphi : W_k(G) \to L^k(G)$ be defined so that $\varphi(u) = K^k(U)$. Then clearly, $\tilde{U} = B^k(\varphi(u))$. We prove that φ is an injective mapping on vertex sets that preserves edges.

First suppose that u and v are distinct vertices in $W_k(G)$. Then U and V are distinct walks in G (although $\tilde{U}$ and $\tilde{V}$ can be identical graphs). Following this, $K(U)$ and $K(V)$ are distinct walks of length $k{-}1$ in $L(G)$, and finally, $K^k(U)$ and $K^k(V)$ are distinct vertices in $L^k(G)$. Hence, φ is injective.

Now suppose that (u, v) is an edge in $W_k(G)$. Then both U and V are walks of length k in G, and their union forms a walk of length $k{+}1$. Assume that $U = (u_0, u_1, \dots, u_k)$, $V = (u_1, u_2, \dots, u_{k+1})$ and denote $Z = (u_0, u_1, \dots, u_{k+1})$. Then $K^{k+1}(Z)$ is a vertex in $L^{k+1}(G)$, and hence, $K^k(Z) = (u', w')$ is an edge in $L^k(G)$. Since $u' = K^k(U)$ and $w' = K^k(V)$, φ preserves edges. □

Let v be a vertex in $L^k(G)$ such that $B^k(v)$ is a path of length k. Then it is easy to prove that there is no other vertex, say u, in $L^k(G)$ such that $B^k(v) = B^k(u)$. Hence, when considering path graphs, there is a unique embedding guaranteed by Theorem 1.

References

[1] Belan A., Jurica P., *Diameter in path graphs*, Acta Math. Univ. Comenianae (to appear).

[2] Broersma H. J., Hoede C., *Path graphs*, J. Graph Theory **13** (1989), 427-444.

[3] Knor M., *A note on the radius of iterated line graphs*, Acta Math. Univ. Comenianae **LXIV, 2** (1995), 255-258.

[4] Knor M., Niepel Ľ., *Connectivity of path graphs*, (submitted).

[5] Knor M., Niepel Ľ., *Distances in iterated path graphs*, (submitted).

[6] Li H., Lin Y., *On the characterization of path graphs*, J. Graph Theory **17** (1993), 463-466.

[7] Niepel Ľ., Knor M., Šoltés Ľ., *Distances in iterated line graphs*, Ars Comb. **43** (1996), 193-202.

[8] Yu X., *Trees and unicyclic graphs with Hamiltonian path graphs*, J. Graph Theory **14** (1990), 705-708.